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The Malyuzhinets theory for scattering from wedge boundaries: a review

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Abstract

The Malyuzhinets technique is reviewed based on his fundamental papers of the 1950s. Subsequent developments are surveyed and recent advances are presented. The review is focused around the basic problem of determining the wave field scattered from the edge of a wedge of exterior angle 2Φ with arbitrary impedance conditions on either face. We begin by establishing a direct relationship between the Sommerfeld integral representation and the Laplace transform. This provides fresh insight into Malyuzhinets' inferences about functions representable via the Sommerfeld integral and, simultaneously, allows us to prove both the inversion formula for the Sommerfeld integral and the crucial nullification theorem. The special functions $\eta_\Phi(z)$ and $\psi_\Phi(z)$ occurring in Malyuzhinets' theory of diffraction from a wedge-shaped region are described. Based on this theoretical background we present a detailed derivation of the well-known Malyuzhinets expressions for the wave field diffracted by an impedance wedge. An alternative representation of the Malyuzhinets solution as a series of Bessel functions is also presented that is completely equivalent to the integral form of the Malyuzhinets solution. This permits a description of the wave field in the vicinity of the edge of an impedance wedge, when $kr \leq 1$, and simple expressions are given for the tip values of the field and its first derivatives. The edge value u_0 can be expressed in terms of Malyuzhinets functions, and its magnitude is easily evaluated if the impedances of the wedge faces are purely imaginary. Thus, $|u_0| \leq \pi/\Phi$ with equality only for a wedge with Neumann boundary conditions. ©1999 Elsevier Science B.V. All rights reserved.

1. Introduction

Diffraction from a wedge is a well covered topic, dating back over a century; see Refs. [1,2] for early citations by Sommerfeld, Poincaré, MacDonald, and others. Corresponding solutions relevant to the Dirichlet or Neumann boundary conditions on the wedge faces are presented in more detail in Refs. [1–9]. The generalised problem with impedance boundary conditions on the faces was solved by Malyuzhinets in his D.Sc. Dissertation [10], and described in a series of classic papers [11–14], culminating in the concise solution outlined in his 1958 paper [15]. His solution was deduced in the form of a Sommerfeld integral with an integrand involving a new special function $\psi_\Phi(z)$. Malyuzhinets later gave a short review of the method in Ref. [16], his only paper published in a non-Russian

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journal. His works extended and transformed the intuitive Sommerfeld approach into an elegant formal procedure that exploits basic concepts of mathematical analysis rather then constructing images of a real source in a fictitious branched Riemann space. Williams [17] independently solved the problem for a wedge with the same impedance on each face in terms of the Sommerfeld integral and a double gamma function, whereas Senior [18] using the Laplace transform provided a solution of an electromagnetic diffraction problem involving a wedge with finite conductivity.

Further contributions to the Malyuzhinets theory were made by Tuzhilin who developed a theory of related functional equations [19–21] and demonstrated the possibility of extending the Malyuzhinets approach to more sophisticated boundary conditions [22] (the solution for the particular case of a thin elastic semi-infinite plate was published in [23–25]). Diffraction of a transient scalar wave by an impedance wedge was considered by Sakharova and Filippov in [26,27], and the result was expressed in terms of the special function $\psi_\Phi(z)$ occurring at the Malyuzhinets theory. In [27] Filippov gave also a complete uniform asymptotic expansion for the far-field scattered by an impedance wedge.

Zavadskii and Sakharova [28] developed the first numerical procedures for computing the Malyuzhinets special function and deduced some useful analytical representations [29,30]. Numerical calculations of the function $\psi_\pi(\alpha)$, relevant to a screen, were discussed by Volakis and Senior [31], and Hongo and Nakajima [32] derived an expansion for the general function $\psi_\Phi(\alpha)$ by using Chebyshev polynomials. Herman et al. [33] provided simple analytic expressions for $\psi_\Phi(\alpha)$ for small and large arguments. Osipov presented more refined approximations accurate to better than 0.01% over the whole complex α plane by combining direct numerical integration in the integral representation of $\psi_\Phi(\alpha)$ given by Zavadskii and Sakharova [29] with summation of a new series representation of the Malyuzhinets function [34]. Further contributions are authored by Aidi and Lavergnat [35] who utilised the results of [34] and also proposed alternative methods permitting to save computational times. The authors of [36] have proposed to evaluate the Malyuzhinets function by rearranging its integral representation and performing numerical integration according to trapezoidal rule, in contrast to Laguerre quadrature employed in [34].

Malyuzhinets' solution, having the form of a Sommerfeld integral, is perfectly suited for the purpose of subsequent far-field analysis. In contrast, the analysis of the near-field behaviour requires an alternative representation for the solution. This was achieved by Budaev and Petrashen' [37] who found series representations of the field diffracted by a wedge with equal face impedances, and by Osipov [38] who deduced the series solution for the general case of arbitrary face impedances.

Depending upon the value of the vertex angle, the model of an impedance wedge uniformly includes a variety of canonical geometries, including an imperfect half-plane, a flat surface with an impedance step, and an impedance horn. Many papers have appeared dealing with both electromagnetic and acoustic applications in these configurations. For instance: plane wave scattering from an impedance half-plane [39,40]; radiation of a line source at the tip of an absorbing wedge [41–46]; Green's functions [47,48]; edge waves [49]; diffraction of surface [45,50,51], plane [45,52–59], cylindrical [45,47,48,60–64], and transient scalar [26,27] waves by an impedance wedge of arbitrary angle. The corresponding mathematical solution for the impedance wedge can therefore serve as a universal basis for treating scattering and diffraction problems. However, the lack of comprehensive publications of Malyuzhinets' results and the apparent complexity of his solution, caused many researchers and engineers to utilise different mathematical approaches which proved to be either less efficient for this class of diffraction problems, such as the Kontorovich–Lebedev transform [65] and the method of eigenfunction expansion [37,66], or fundamentally restricted to rectangular geometries, such as the Wiener–Hopf technique [67], or the methods described in [68–71]. In this paper we restrict ourselves to the Malyuzhinets method and the arbitrary-angled impedance wedge, and therefore this review does not include a great many publications dealing with the Wiener–Hopf method and rectangular geometries (for a detailed discussion of this subject see [72–74]).

The purpose of this paper is not to derive the solution for the impedance wedge, which is well known, but to discuss the key points of the Malyuzhinets theory, emphasizing its fundamental character and the physical clarity of its consequences. We believe this is important for further understanding of wave phenomena because Malyuzhinets' approach provides a powerful means for tackling one of the basic questions of wave theory, that is, the diffraction by edged obstacles. It should be pointed out that this paper gives our own insights into this theory, which almost certainly differs from the original and highly unconventional concepts introduced by Malyuzhinets, which basically utilised sophisticated geometrical constructions rather than algebraic manipulations [10]. This review is intended to demonstrate that the key principles of Malyuzhinets' method can be explained in a simple way, easily understandable by Wiener–Hopf people, in terms of certain well-known facts from the theory of functions of a complex variable. We hope that this will satisfy the demand for more detailed explanations of the method that has arisen decades after the publication of Malyuzhinets' famous papers of the 50's (see, for example, [75]).

The paper is organised as follows. Section 2 is devoted to foundations of the Malyuzhinets theory. It starts (Section 2.1) with two key propositions in Malyuzhinets' method: the inversion formula for the Sommerfeld integral and the nullification theorem [13]. Unlike Malyuzhinets' original papers, this section presents an alternative derivation of these identities which makes use of the relationship between the Sommerfeld and Laplace integral representations. This allows us to interpret the foundations of the Malyuzhinets method as a direct consequence of the Watson lemma and the Liouville theorem, the two basic statements of the Laplace transform theory. In this section we follow the approach presented in [76]. The properties of the Sommerfeld integral are also discussed in [45,66].

Section 2.2 describes a special function $\eta_\Phi(z)$ relevant to the solution of the radiation problem when the wave field is excited within a wedge-shaped region by sources distributed on its boundaries [11,12]. Section 2.3 is devoted to the special function $\psi_\Phi(z)$ that arises from the theory of diffraction by a wedge with impedance boundary conditions [15]. These two sections gather together almost all the known analytical results concerning the functions $\eta_\Phi(z)$ and $\psi_\Phi(z)$ presented in the literature so far.

Section 3 addresses the derivation of Malyuzhinets' solution for diffraction from an impedance wedge. We provide a detailed and complete derivation of the relevant transform function, using only the Fourier transformation. This section may be considered as our interpretation of the procedure only briefly outlined in the famous Malyuzhinets paper [15]. Section 3 concludes with a closed-form expression for the transform function in terms of the special function ψ_Φ introduced in Section 2.3.

Section 4 of the paper deals with another form of the solution for an impedance wedge. We show in Section 4.1 how the Malyuzhinets solution deduced initially as a Sommerfeld integral can be transformed into a series in terms of Bessel functions. We present an alternative representation which is exact and completely equivalent to the initial Sommerfeld integral, and in agreement with the well-known series solution for wedges with ideal boundaries.

The series form of the solution has some advantages over the Sommerfeld integral for analysing the wave field in the near-and intermediate zones where kr is no longer a large parameter. On the basis of the series representation we deduce simple expressions showing how the singular components of the wave field behave close to the edge of an impedance wedge.

In Section 4.2 we study the tip value of the field, providing both computational results and simple analytical formulae. These results have proven useful in the far-field analysis of the Malyuzhinets solution.

Because of the space limitations, this paper does not address topics related to the far-field analysis of the Malyuzhinets solution. This is the subject of a separate paper [77], which discusses the far-field specifically and provides some new results for the diffraction coefficient.

2. The foundations

2.1. The Sommerfeld integral and its inversion

We are concerned with solutions to the Helmholtz equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} + k^2 u = 0, \tag{1}$$

within a wedge-shaped region $0 < r < \infty$, $|\phi| \le \Phi$, see Fig. 1. Here $k = \omega/c$ and c is the wave speed. The

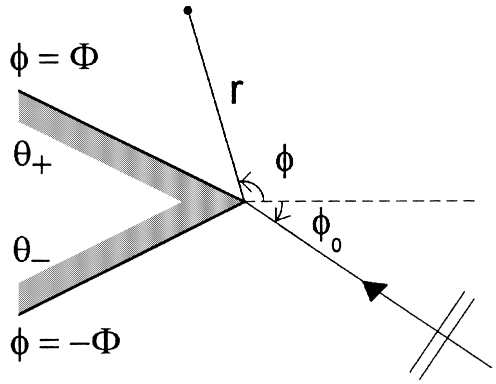

Fig. 1. The geometry.

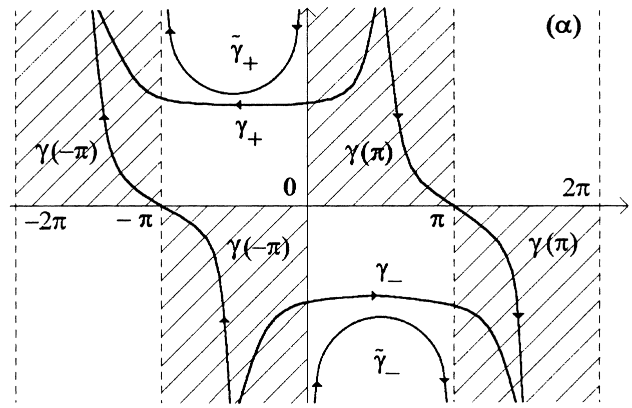

Fig. 2. Integration contours of the Sommerfeld integral.

function $u(r, \phi)$ represents either the sound pressure in acoustics or a component of the electric/magnetic field in electromagnetics.

Within the framework of the Malyuzhinets method the solution to the problem of diffraction by a wedge-shaped domain is sought in the form of an integral

$$u(r, \phi) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma} e^{-\mathrm{i}kr\cos\alpha} S(\alpha + \phi)\, \mathrm{d}\alpha, \tag{2}$$

taken over the contour $\gamma = \gamma_+ \bigcup \gamma_-$ in the spectral complex plane α (Fig. 2). Here γ_+ is a loop in the upper half of the complex α-plane, beginning at $\pi/2 + \mathrm{i}\infty$, ending at $-3\pi/2 + \mathrm{i}\infty$, with $\mathrm{Im}\,\alpha$ lying above an arbitrary minimum, such that no singularities of the integrand occur within γ_+ for all $|\phi| \le \Phi$. The contour γ_- is the image of γ_+ under inversion about the origin $\alpha = 0$. The ends of the integration contours are located in those portions of the complex α-plane (hatched in Fig. 2) where $\mathrm{Im}(k\cos\alpha) < 0$ to ensure convergence of the integral.

The integral of the form (2) has been introduced by Sommerfeld in his famous paper of 1896 on diffraction of an electromagnetic wave from a perfectly conducting half-plane. For arbitrary spectral function $S(\alpha)$ the Sommerfeld integral (2) satisfies the Helmholtz equation (1) and can be interpreted as an expansion of the wave field into a plane wave spectrum.

The symmetry of γ_+ and γ_- yields

$$u(r, \phi) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma_+} \mathrm{e}^{-\mathrm{i}kr\cos\alpha} [S(\alpha + \phi) - S(-\alpha + \phi)]\, \mathrm{d}\alpha, \tag{3}$$

implying that the integral (2) is invariant under the transformation $S(\alpha) \to S(\alpha) + \text{const}$. By appropriate choice of this constant we can set, without loss of generality,

$$S(\mathrm{i}\infty) = -S(-\mathrm{i}\infty), \tag{4}$$

which will be assumed hereinafter. This limiting value of S in (4) is related to the potential function at the wedge tip. Thus, as $r \to 0$ we may let the contour γ_+ recede towards $\mathrm{i}\infty$, in the limit obtaining simply

$$u(0, \phi) = 2\mathrm{i}S(\mathrm{i}\infty). \tag{5}$$

The Sommerfeld integral representation (3) can be expressed in a concise form

$$F(r) = \frac{1}{\pi \mathrm{i}} \int_{\gamma_+} \mathrm{e}^{-\mathrm{i}kr\cos\alpha} f(\alpha)\,\mathrm{d}\alpha, \tag{6}$$

with $F(r) = u(r, \varphi)$ and $2f(\alpha) = S(\alpha + \varphi) - S(-\alpha + \varphi)$, representing at a given value of the parameter φ the potential function and the odd part of the transform function, respectively. Thus, the key point of the Malyuzhinets theory is to prove that the functions $F(r)$ to be sought can indeed be represented as (6) with $f(\alpha)$ being an odd function of α, analytic inside the contour γ_+ and bounded at infinity $\operatorname{Im}\alpha = +\infty$.

To this end, deform the integration contour γ_+ into a new contour $\tilde{\gamma}_+$ (Fig. 2) lying entirely within the strip on the complex α-plane in which $\operatorname{Im}(k\cos\alpha) > 0$, and which is defined by the equation $\operatorname{Re}(-ik\cos\alpha) = \sigma$, where σ is a positive constant. Along this latter contour the exponent function in (6) is oscillating and bounded, which ensures the convergence of the integral so far as the bounded transform functions $f(\alpha)$ are concerned. Notice that no singular points of the function $f(\alpha)$ may fall between the contours γ_+ and $\tilde{\gamma}_+$ because of analyticity of $f(\alpha)$ inside the loop γ_+. Thus, one has

$$F(r) = \frac{1}{\pi \mathrm{i}} \int_{\tilde{\gamma}_+} \mathrm{e}^{-\mathrm{i}kr\cos\alpha} f(\alpha)\,\mathrm{d}\alpha. \tag{7}$$

Next, changing the integration variable according to $p = -\mathrm{i}\cos\alpha$ transforms the expression (7) into

$$F(r) = \frac{1}{2\pi \mathrm{i}} \int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} \mathrm{e}^{pr} Q(p)\,\mathrm{d}p, \tag{8}$$

with

$$Q(p) = \frac{2f(\alpha)}{\mathrm{i}k\sin\alpha}. \tag{9}$$

Eq. (8) represents an inverse Laplace transform, and its inversion can therefore be achieved by the direct Laplace transform

$$Q(p) = \int_0^{+\infty} \mathrm{e}^{-pr} F(r)\,\mathrm{d}r, \tag{10}$$

or, in terms of the variable α,

$$f(\alpha) = \frac{\mathrm{i}k}{2}\sin\alpha \int_0^{+\infty} \mathrm{e}^{\mathrm{i}kr\cos\alpha} F(r)\,\mathrm{d}r. \tag{11}$$

Expressions (7) and (11) are completely equivalent to those of the Laplace transform, (8) and (10), respectively. The analyticity of the transform function $f(\alpha)$ inside the contour $\tilde{\gamma}_+$ results from that of the Laplace transform function $Q(p)$ to the right of the contour $\operatorname{Re} p = \sigma$ [78]. The fact that $f(\alpha)$ is an odd function of α is dictated by the form of Eq. (11). Indeed, for $F(r) = O[\exp(-br)]$ with $b > 0$ when $r \to +\infty$, which is true for outgoing

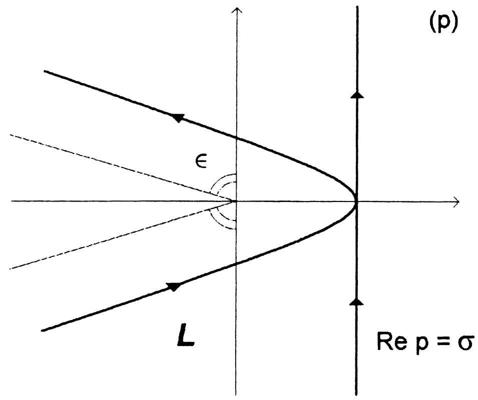

Fig. 3. Integration contours in the complex p plane.

fields in media with absorption, arbitrarily small at least, the exponent $\exp(\mathrm{i}kr\cos\alpha)$ in (11) can be expanded into a power series in $\mathrm{i}kr\cos\alpha$, thus proving the oddness of $f(\alpha)$.

The asymptotic behaviour of the transform function $f(\alpha)$ as $\mathrm{Im}\,\alpha \to +\infty$ can be derived using Watson's lemma, one of the key statements in the theory of the Laplace transform, which relates the behaviour of the original function $F(r)$ for $r \to 0$ to that of the image function $Q(p)$ when $|p| \to +\infty$ [78]:

Lemma 1 (Watson). *Let* δ *be a non-negative constant,* $F(r)$ *an analytic function on the complex* r*-plane within a sector* $R_\delta = \{r : |\arg r| \le \delta, 0 < |r| < +\infty\}$ *in which it satisfies uniformly in* $\arg r$ *the inequality*

$$|F(r)| < M|r|^{-1+a}\exp(b|r|), \tag{12}$$

with M, a, b *being certain real and positive numbers. Then, for* $|p| \to +\infty$ *in the sector* $P_\delta = \{p : |\arg p| < \pi/2 + \delta, 0 < |p| < +\infty\}$ *the following estimate is true*

$$|Q(p)| = O(|p|^{-a}), \tag{13}$$

where $Q(p)$ *is the Laplace image function associated with the original function* $F(r)$.

It follows from this lemma and the formula (9) that on the complex α-plane in the strip $\arg k - \pi - \delta < \mathrm{Re}\,\alpha < \arg k + \delta$ when $\mathrm{Im}\,\alpha \to +\infty$ the Sommerfeld image function is estimated as $|f(\alpha)| = \mathrm{O}\{\exp[(1-a)\mathrm{Im}\,\alpha]\}$. Specifically, $f(\alpha)$ can be bounded by a constant if the original function $F(r)$ takes a finite value at the point $r = 0$, which corresponds to $a = 1$ in the expression (12).

In order to complete the derivation of the Malyuzhinets inversion formula for the Sommerfeld integral taken over the wide contour γ_+, see Eq. (6), one should justify widening the narrow integration contour $\tilde{\gamma}_+$ in Eq. (7) into the crosshatched regions on the complex α-plane (Fig. 2). In the complex p-plane this corresponds to replacing the contour Re $p = \sigma$ with a new one, say L, going to infinity in the left half-plane along the directions $\arg p = \pm(\pi/2 + \epsilon)$ with $0 < \epsilon < \pi/2$ (Fig. 3).

One can readily see that the constraints placed on the original function $F(r)$ by the Watson lemma are sufficient to validate such a contour deformation within the sector P_δ for large enough values of $|p|$. Indeed, it follows from the theory of the Laplace transform [78] that for sufficiently large values of $|p|$ (with a possible exception of the point at infinity $|p| = \infty$) the image function $Q(p)$ has no singular points in the sector P_δ if the original function $F(r)$ in the neighbourhood of the point $|r| = +\infty$ in the sector R_δ is analytic and meets the estimate

$$|F(r)| < M_1\exp(b_1|r|), \tag{14}$$

where M_1, b_1 are certain positive constants. The last inequality (14) is automatically satisfied by functions complying with condition (12) of the Watson lemma. Thus, each function $F(r)$ conforming to the estimate (12) has as its Laplace image a function $Q(p)$ which is both vanishing according to (13) and analytic within the sector P_δ for $|p| \to +\infty$ (note that $Q(p)$ may have singular points in the sector P_δ, but only at finite distances from the origin $p = 0$).

It is now straightforward to formulate the Malyuzhinets theorem on the inversion of the Sommerfeld integral [23]. In fact, this theorem results from the foregoing as a direct consequence of the Watson lemma.

Theorem 2 (Inversion formula for the Sommerfeld integral). *Let M, M_1, a, b, δ be positive numbers, and let ϵ be a number satisfying $0 < \epsilon < \inf(\delta, \pi)$. Let $F(r)$ be an analytic function in the entire sector R_δ in which it satisfies uniformly in* arg *r the inequality* (12):

$$|F(r)| < M|r|^{-1+a}\exp(b|r|).$$

Consider the Sommerfeld integral (6) *over the contour γ_+ that goes from $\alpha = \arg k + \epsilon + i\infty$ to $\alpha = \arg k - \pi - \epsilon + i\infty$. Then, among odd functions $f(\alpha)$, which are analytic on and within the contour γ_+ except at infinity, and which may not grow like* $\exp(\mathrm{Im}\,\alpha)$ *or faster as* $\mathrm{Im}\,\alpha \to +\infty$,

(i) *there exists one and only one solution $f(\alpha)$ to the integral equation* (6);

(ii) *for* $\mathrm{Im}(k\cos\alpha) > b$ *this solution is represented by the inversion formula* (11);

(iii) *inside the contour γ_+ when* $\mathrm{Im}\,\alpha \to +\infty$ *it is estimated by the inequality*

$$|f(\alpha)| < M_1 \exp[(1-a)\mathrm{Im}\,\alpha].$$

An important consequence of this theorem is that there are functions $F(r)$ that can be represented as the Sommerfeld integral (7) over the narrow contour $\tilde{\gamma}_+$ but not if the integration is performed over the wide contour γ_+. For example, solutions of wave problems involving point sources, like the Green function, possess a singularity at the source location point and therefore can not be represented as the conventional integral (6) which becomes in this case divergent. However, it is still possible to use the modified integral (7) to represent such solutions (see [48]).

To conclude our discussion on the foundations of the Malyuzhinets theory, let us take a look at another key statement of this theory: the nullification theorem for the Sommerfeld integral [13].

Theorem 3 (Nullification of the Sommerfeld integral). *Consider the homogeneous integral equation*

$$\frac{1}{\pi \mathrm{i}}\int_{\gamma_+} \mathrm{e}^{-\mathrm{i}kr\cos\alpha} f(\alpha)\,\mathrm{d}\alpha = 0. \tag{15}$$

Let the asymptotic behaviour of the function $f(\alpha)$ when $\mathrm{Im}\,\alpha \to +\infty$ *be bounded by the estimate* $|f(\alpha)| \le \mathrm{O}[\exp(D\,\mathrm{Im}\,\alpha)]$ *where D denotes a real number, positive or negative. Then, among odd functions $f(\alpha)$, analytic on and within the contour γ_+ except at the point at infinity,*

(i) *there exists only the trivial solution $f(\alpha) \equiv 0$ to the* Eq. (15) *if $D \in (-\infty, 1)$;*

(ii) *otherwise, for $D \ge 1$ the homogeneous integral equation* (15) *is satisfied by any trigonometric polynomial expression of the form $f(\alpha) = \sin\alpha \sum_{m=1}^{n} C_m \cos^{m-1}\alpha$ where C_m are arbitrary constants, and n means the integer part of D.*

This theorem is used to solve equations that arise from applying the Sommerfeld integral to boundary conditions involving higher-order field derivatives with respect to the space coordinates. The examples are the boundary conditions simulating the presence of a membrane or a thin elastic plate in an acoustic media, or a thin non-metallic sheet in problems of electromagnetic diffraction.

Malyuzhinets proved the nullification theorem by integrating by parts in Eq. (15), thus reducing (15) to a form which permits the application of the inversion formula with $F(r) \equiv 0$. It is shown below that the nullification theorem follows from the so-called extended Liouville theorem (see, for example, [79], p. 84).

Consider Eq. (15) in terms of the complex variable $p = -\mathrm{i}k\cos\alpha$, related to the Laplace transform. Then one has the equation

$$\int_L \mathrm{e}^{pr} Q(p)\mathrm{d}p = 0, \quad r > 0, \tag{16}$$

in which the contour L is the image of the contour γ_+ in the complex p-plane (Fig. 3). Eq. (16) implies that $Q(p)$ has no singular points inside the contour L. On the other hand by the definition of the contour γ_+ the function $Q(p)$ must be analytic to the right of the contour L as well, since this area corresponds to the interior of γ_+. Thus, the function $Q(p)$ is an entire function of the complex variable p over the whole complex p-plane.

If the asymptotic behaviour of $f(\alpha)$ is estimated with $D < 1$, then according to its definition in Eq. (9) $Q(p) \to 0$ as $|p| \to +\infty$. The further application of Liouville's theorem, which states that a bounded entire function is a constant, uniquely determines $Q(p)$ to be identically zero. Alternatively, if $D \geq 1$, then in the vicinity of the point at infinity the function $Q(p)$ behaves like $\mathrm{O}(|p|^{D-1})$, and according to the extended Liouville theorem the function $Q(p)$ is a polynomial in p of degree not exceeding the integer part of the difference $D-1$. In terms of the variable α this immediately gives $f(\alpha) = \sin\alpha \sum_{m=1}^{n} C_m \cos^{m-1}\alpha$, which completes the proof of the nullification theorem.

2.2. The function η_Φ

We begin with the function η_Φ introduced by Malyuzhinets in 1955 [11] for solving the problem of acoustic radiation from the faces of a wedge undergoing prescribed normal velocity. The normal velocity into the fluid on either face is

$$v = \pm \frac{\mathrm{i}}{\omega\rho r} \left.\frac{\partial u}{\partial\phi}(r,\phi)\right|_{\phi=\pm\Phi},$$

where ρ is the fluid density. Therefore, we first consider the boundary condition

$$\begin{aligned} &\frac{\mathrm{i}}{kr}\frac{\partial u}{\partial\phi}(r,\phi) = 1, \quad \phi = \Phi, \ \ 0 < r < \infty, \\ &\frac{\mathrm{i}}{kr}\frac{\partial u}{\partial\phi}(r,\phi) = 0, \quad \phi = -\Phi, \ \ 0 < r < \infty. \end{aligned} \tag{17}$$

We begin by noting that the Sommerfeld transform of $F = 1$ is $f(\alpha) = -\frac{1}{2}\tan\alpha$, or

$$\frac{1}{2\pi\mathrm{i}}\int_\gamma e^{-\mathrm{i}kr\cos\alpha}\tan\alpha\,\mathrm{d}\alpha = -2, \tag{18}$$

which can be deduced by first using Eq. (3) followed by the change of variable $\cos\alpha = x$. This permits us to rewrite the boundary conditions (17) in the form

$$\begin{aligned} &\frac{1}{2\pi\mathrm{i}}\int_{\gamma_+} \mathrm{e}^{-\mathrm{i}kr\cos\alpha}[S(\alpha+\Phi) + S(-\alpha+\Phi) + \sec\alpha]\sin\alpha\,\mathrm{d}\alpha = 0, \\ &\frac{1}{2\pi\mathrm{i}}\int_{\gamma_+} \mathrm{e}^{-\mathrm{i}kr\cos\alpha}[S(\alpha-\Phi) + S(-\alpha-\Phi)]\sin\alpha\,\mathrm{d}\alpha = 0. \end{aligned} \tag{19}$$

According to the nullification theorem, the integrals in Eq. (19) vanish for all $r > 0$ if and only if the integrand functions vanish for all α. Putting $S(\alpha) = \eta_\Phi(\alpha+\Phi)$ yields the pair of functional equations

$$\eta_\Phi(\alpha+2\Phi) + \eta_\Phi(-\alpha+2\Phi) = -\sec\alpha, \qquad \eta_\Phi(\alpha) + \eta_\Phi(-\alpha) = 0, \tag{20}$$

the latter implying that $\eta_\Phi(\alpha)$ is an odd function of its argument. The former equation may therefore be written as

$$\eta_\Phi(\alpha-2\Phi) - \eta_\Phi(\alpha+2\Phi) = \sec\alpha. \tag{21}$$

The equation in (21) can be solved using the integral transformation [15]

$$\eta_\Phi(\alpha) = \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{e}^{-\mathrm{i}t\alpha} G(t)\,\mathrm{d}t, \qquad G(t) = -\frac{1}{2\pi}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{e}^{\mathrm{i}t\alpha}\eta_\Phi(\alpha)\,\mathrm{d}\alpha, \tag{22}$$

which differs from the conventional Fourier transform only by a change of variables. Multiplying both sides of Eq. (21) by $\exp(\mathrm{i}t\alpha)$ and integrating, yields

$$G(t) = \frac{-1}{4\pi \mathrm{i} \sin(2\Phi t)} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathrm{e}^{\mathrm{i}t\alpha}}{\cos\alpha} \mathrm{d}\alpha = -\left[4 \sin(2\Phi t) \cos\left(\frac{\pi t}{2}\right)\right]^{-1}, \tag{23}$$

and therefore,

$$\eta_\Phi(\alpha) = -\frac{1}{4} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{e}^{-\mathrm{i}t\alpha}}{\sin(2t\Phi)\cos(\pi t/2)} \mathrm{d}t. \tag{24}$$

This is the fundamental identity which we will use later for the problem with impedance boundary conditions.

Eq. (24) may be rewritten [29]

$$\eta_\Phi(\alpha) = -\frac{1}{2} \int_0^\infty \frac{\sinh(s\alpha)\mathrm{d}s}{\cosh(\frac{1}{2}\pi s)\sinh(2\Phi s)}. \tag{25}$$

The function η_Φ has the additional properties

$$\eta_\Phi\left(\alpha + \frac{\pi}{2}\right) + \eta_\Phi\left(\alpha - \frac{\pi}{2}\right) = -\frac{\pi}{4\Phi} \tan\left(\frac{\pi\alpha}{4\Phi}\right), \tag{26}$$

$$\eta_\Phi(\alpha + \Phi) + \eta_\Phi(\alpha - \Phi) = \eta_{\Phi/2}(\alpha), \tag{27}$$

which follow from (25) and the integral identities, valid for $|\mathrm{Re}\,\alpha| < 1$,

$$\int_0^\infty \frac{\cosh(t\alpha)}{\cosh t} \mathrm{d}t = \frac{\pi}{2} \sec\left(\frac{\pi\alpha}{2}\right), \qquad \int_0^\infty \frac{\sinh(t\alpha)}{\sinh t} \mathrm{d}t = \frac{\pi}{2} \tan\left(\frac{\pi\alpha}{2}\right). \tag{28}$$

Eq. (25) implies that $\eta_\Phi(\alpha)$ is analytic for $|\mathrm{Re}\,\alpha| < \frac{1}{2}\pi + 2\Phi$. It may be continued to values of α outside of this by repeated use of the functional relation (21). Therefore, $\eta_\Phi(\alpha)$ possesses only simple poles and no branch cuts, and the poles are at the points

$$\alpha = \pm\alpha_{nm}, \quad \alpha_{nm} = \frac{\pi}{2}(2m-1) + 2\Phi(2n-1), \tag{29}$$

for $n, m = 1, 2, 3, \ldots$, with residues $(-1)^{m-1}$, implying [11]

$$\eta_\Phi(\alpha) = \sum_{l=1}^\infty \sum_{q=1}^\infty \frac{(-1)^{l+1} 2\alpha}{\alpha^2 - [\frac{\pi}{2}(2l-1) + 2\Phi(2q-1)]^2}. \tag{30}$$

This simplifies further when $\Phi = n\pi/(4m)$, where n/m is rational and irreducible: thus [15],

$$\eta_{n\pi/4m}(\alpha) = \frac{1}{n} \sum_{k=1}^m \sum_{l=1}^n (-1)^l \frac{1}{2} \tan\left[\frac{\alpha}{2n} + \frac{1}{2} a(k,l)\right], \quad n \text{ odd},$$

$$\eta_{n\pi/4m}(\alpha) = \frac{1}{n} \sum_{k=1}^m \sum_{l=1}^n (-1)^l \frac{1}{\pi} \left[\frac{\alpha}{n} + a(k,l)\right] \cot\left[\frac{\alpha}{n} + a(k,l)\right], \quad n \text{ even}, \tag{31}$$

where

$$a(k,l) = \frac{\pi}{2}\left(\frac{2l-1}{n} - \frac{2k-1}{m}\right). \tag{32}$$

For example [11,32],

$$\eta_{\pi/4}(\alpha) = -\frac{1}{2}\tan\frac{\alpha}{2}, \qquad \eta_{\pi/2}(\alpha) = \frac{2\alpha - \pi\sin\alpha}{4\pi\cos\alpha},$$
$$\eta_{3\pi/4}(\alpha) = -\frac{1}{6}\tan\left(\frac{\alpha}{6}\right)\left(\frac{3+2\cos(\alpha/3)}{1+2\cos(\alpha/3)}\right),$$
$$\eta_{\pi}(\alpha) = \frac{(\sqrt{2} - \cos(\alpha/2))\sin(\alpha/2) - \alpha/\pi}{4\cos\alpha}.$$

Returning to the original problem, we can now consider the case when the faces each have constant prescribed normal velocities $v_\pm$ on $\phi = \pm\Phi$. The solution is given by (2) with

$$S(\alpha) = \rho c v_+ \eta_\Phi(\alpha + \Phi) + \rho c v_- \eta_\Phi(\alpha - \Phi).$$

The value of $\eta_\Phi(\mathrm{i}\infty)$ is readily deduced from (26) as $\eta_\Phi(\mathrm{i}\infty) = -\mathrm{i}\pi/(8\Phi)$, which together with (5) implies that the wave field at the edge, the tip pressure, is [11]

$$u(0,\phi) = \frac{\pi}{4\Phi}\rho c(v_+ + v_-). \tag{33}$$

Malyuzhinets [12] derived further results concerning the solution for vibrating faces. Among them is the remarkable fact that the acoustic power radiated from the faces in the region $0 < r < r_1$ is

$$\frac{\rho c}{2}(|v_+|^2 + |v_-|^2)r_1,$$

for large values of r_1. This is the same power predicted on the basis of the plane wave approximation $u(r,\pm\Phi) = \rho c v_\pm$, and implies that the edge causes no additional radiation. Malyuzhinets also considered the more general situation where the faces vibrate with velocity proportional to $\exp(-\mathrm{i}kr\cos\beta)$, β constant. He introduced and discussed the generalised function $\eta^*_\Phi(\alpha,\beta)$ [12] which reduces to $\eta_\Phi(\alpha)$ when $\beta = \frac{1}{2}\pi$.

2.3. The function ψ_Φ

The Malyuzhinets function ψ_Φ [15] is closely related to η_Φ mathematically, although the physical interpretation is not as immediate. We will return to this later, but for the moment define ψ_Φ as

$$\psi_\Phi(\alpha) = \exp\left[\int_0^\alpha \eta_\Phi(t)\,\mathrm{d}t\right]. \tag{34}$$

It is an even function of its argument and it follows from Eqs. (25) and (34) on carrying out the integration with respect to t that [29]

$$\psi_\Phi(\alpha) = \exp\left[-\frac{1}{2}\int_0^{+\infty}\frac{\cosh(t\alpha) - 1}{t\cosh(t\pi/2)\sinh(2t\Phi)}\mathrm{d}t\right], \tag{35}$$

$|\mathrm{Re}\,\alpha| < \frac{1}{2}\pi + 2\Phi$. It may be continued outside this strip by using any of the functional properties [15]:

$$\frac{\psi_\Phi(\alpha + 2\Phi)}{\psi_\Phi(\alpha - 2\Phi)} = \cot\left(\frac{\alpha}{2} + \frac{\pi}{4}\right), \tag{36}$$

$$\psi_\Phi\left(\alpha + \frac{\pi}{2}\right)\psi_\Phi\left(\alpha - \frac{\pi}{2}\right) = \psi_\Phi^2\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi\alpha}{4\Phi}\right), \tag{37}$$

$$\psi_\Phi(\alpha + \Phi)\psi_\Phi(\alpha - \Phi) = \psi_\Phi^2(\Phi)\psi_{\Phi/2}(\alpha). \tag{38}$$

These are immediate consequences of Eqs. (21) and (26), and the identity

$$\int_0^\alpha \sec z \mathrm{d}z = \log \tan\left(\frac{\alpha}{2} + \frac{\pi}{4}\right). \tag{39}$$

Moreover, $\psi_\Phi(\bar{\alpha}) = \bar{\psi}_\Phi(\alpha)$ where the bar denotes a complex conjugate, and this together with its evenness property and the functional relations (36) and (37) implies that $\psi_\Phi(\alpha)$ can be found for any complex-valued α once it is known in the fundamental domain $0 \le \mathrm{Re}\,\alpha \le \inf(\frac{1}{2}\pi, 2\Phi)$, $\mathrm{Im}\,\alpha \ge 0$.

As a function of the complex variable α the Malyuzhinets function $\psi_\Phi(\alpha)$ has zeros at $\alpha = \pm\alpha_{nm}$ for $n = 1, 2, 3, \dots, m = 1, 3, 5, \dots$ and poles for $m = 2, 4, 6, \dots$. Therefore, using Eqs. (30) and (34)

$$\psi_\Phi(\alpha) = \prod_{l=1}^{\infty}\prod_{k=1}^{\infty}\left\{1 - \frac{\alpha^2}{[\frac{\pi}{2}(2l-1) + 2\Phi(2k-1)]^2}\right\}^{(-1)^{l+1}}. \tag{40}$$

This again simplifies for $\Phi = \pi n/(4m)$. Using Eq. (31), we find that [15],

$$\psi_{\pi n/4m}(\alpha) = \prod_{k=1}^{m}\prod_{l=1}^{n}\left(\frac{\cos[\frac{1}{2}a(k,l)]}{\cos[\frac{1}{2}a(k,l) + \frac{\alpha}{2n}]}\right)^{(-1)^l}, \quad \text{if } n \text{ is odd},$$

$$\psi_{\pi n/4m}(\alpha) = \prod_{k=1}^{m}\prod_{l=1}^{n}\exp\left[\frac{(-1)^l}{\pi}\int_{a(k,l)}^{a(k,l)+\alpha/n} u \cot u \mathrm{d}u\right], \quad \text{if } n \text{ is even}, \tag{41}$$

where $a(k,l)$ are defined in (32). Specific examples are [15],

$$\psi_{\pi/4}(\alpha) = \cos\frac{\alpha}{2}, \qquad \psi_{3\pi/4}(\alpha) = \frac{4}{3}\cos\frac{\alpha}{6} - \frac{1}{3}\sec\frac{\alpha}{6},$$

$$\psi_{\pi/2}(\alpha) = \exp\left(\frac{1}{4\pi}\int_0^\alpha \frac{2t - \pi\sin t}{\cos t}\mathrm{d}t\right),$$

$$\psi_\pi(\alpha) = \exp\left[-\frac{1}{8\pi}\int_0^\alpha \frac{\pi\sin t - 2\sqrt{2}\pi\sin(t/2) + 2t}{\cos t}\mathrm{d}t\right].$$

The Malyuzhinets function can also be represented as follows [34,38]:

$$\psi_\Phi(z) = \frac{1}{\sqrt{2}}\psi_\Phi\left(\frac{\pi}{2}\right)\exp\left\{-\mathrm{i}s\frac{\pi z}{8\Phi} + I(sz, \Phi)\right\}. \tag{42}$$

Here $s = \mathrm{sign}(\mathrm{Im}\,z)$ and

$$I(w, \Phi) = \sum_{k=1}^{\infty}(-1)^{k+1}\left\{\frac{\mathrm{e}^{\mathrm{i}\pi kw/(2\Phi)}}{2k\cos[\pi^2 k/(4\Phi)]} + \frac{\mathrm{e}^{\mathrm{i}(2k-1)w}}{(2k-1)\sin[2\Phi(2k-1)]}\right\}. \tag{43}$$

The series in Eq. (43) converges absolutely if $\mathrm{Im}\,w > 0$. For $|\mathrm{Im}\,z| \gg 1$ the term $I(w, \Phi)$ in Eq. (42) becomes negligibly small, which leads to the asymptotic formula

$$\psi_\Phi(z) = \frac{1}{\sqrt{2}}\psi_\Phi\left(\frac{\pi}{2}\right)\exp\left(-\mathrm{i}s\frac{\pi z}{8\Phi}\right). \tag{44}$$

The latter coincides asymptotically with the one given by Malyuzhinets [16]

$$\psi_\Phi(z) \approx \sqrt{\cos\left(\frac{\pi z}{4\Phi}\right)}\exp\left\{-\frac{1}{2\pi}\int_{-\infty}^{+\infty}\ln\left[\mathrm{ch}\left(\frac{\pi t}{4\Phi}\right)\right]\frac{\mathrm{d}t}{\mathrm{ch}\,t}\right\}, \tag{45}$$

and also matches the more complicated expressions deduced in [26,30]. When $\Phi = \pi n/[2(2m-1)]$ with n and m integer numbers, certain members of the series (43) become infinite. Nevertheless, these singularities cancel each other and the total remains bounded.

A FORTRAN program for the computation of $\psi_\Phi(\alpha)$ using the approximations from [33] is available in the book [73].

3. Malyuzhinets' solution for diffraction from an impedance wedge

We consider the same acoustic configuration as in Section 2.2 where the faces of the wedge now have impedance boundary conditions of the form

$$u(r, \pm\Phi) + Z_\pm v_\pm(r) = 0, \quad 0 < r < \infty, \tag{46}$$

for complex-valued constants $Z_\pm$. The excitation is assumed to come from an incident plane wave from the direction ϕ_0,

$$u_{\mathrm{inc}}(r, \phi) = U_0 \exp[-\mathrm{i}kr\cos(\phi - \phi_0)]. \tag{47}$$

The problem differs from that of the vibrating faces because now both the surface pressure and normal velocity are unknowns. The parameters $Z_\pm$ are the specific acoustic impedances $Z_\pm$ [2] of the faces and are assumed to have non-negative real parts. It is simpler to work with the complex-valued angles $\theta_\pm$ defined by

$$\sin\theta_\pm = Z_0/Z_\pm, \tag{48}$$

which have $0 < \mathrm{Re}\,\theta_\pm \le \pi/2$ and arbitrary imaginary parts (here Z_0 is the free space impedance, ρc in acoustics). The impedance boundary conditions thus reduce to the following conditions for the pressure

$$\frac{\mathrm{i}}{kr}\frac{\partial u}{\partial\phi} \pm \sin\theta_\pm\, u = 0, \phi = \pm\Phi. \tag{49}$$

Malyuzhinets' theory can also handle complex valued incidence angles ϕ_0, which allows one to consider excitation arising from non-homogeneous incident fields. In particular, putting $\phi_0 = \Phi - \theta_+$ with $\mathrm{Im}\,\theta_+ < 0$ in Eq. (47) gives a surface wave travelling along the upper face of the wedge towards its edge. Analogously, the substitution $\phi_0 = -\Phi + \theta_-$ with $\mathrm{Im}\,\theta_- < 0$ transforms the excitation (47) into an incoming surface wave propagating over the lower face of the wedge. For $\mathrm{Im}\,\theta_\pm > 0$ these are no longer surface waves because their amplitudes do not decay as the observation point moves away from the boundaries. Note also that the function (47) describes an *incoming* wave, that is, one going from infinity toward the edge if $|\mathrm{Re}\,\phi_0| < \Phi$. Thus, in what follows we assume that the parameter ϕ_0 can be a complex number with $-\Phi < \mathrm{Re}\,\phi_0 < \Phi$ and an arbitrary imaginary part, that is, $-\infty < \mathrm{Im}\,\phi_0 < +\infty$.

Similar mathematical problems appear in electromagnetics in the case that the plane wave is incident transverse to the edge of the wedge. Then, assuming that the edge is aligned with the z axis of the cylindrical coordinate system (r, ϕ, z), one has $u(r, \phi) = H_z(r, \phi)$ and $\sin\theta_\pm = \eta_\pm$ for H-polarization and $u(r, \phi) = E_z(r, \phi)$ and $\sin\theta_\pm = \eta_\pm^{-1}$ for E-polarization where H_z and E_z are the z components of the magnetic and electric fields, respectively, and $\eta_\pm$ stand for the normalised surface impedances [73].

We proceed as before, assuming $u(r, \phi)$ has the form of the Sommerfeld integral (2), subject to the condition that the far-field must now reduce to the given excitation (47)). Applying the boundary conditions (49) to u given by Eq. (2) yields two integral identities

$$\int_\gamma \mathrm{e}^{-\mathrm{i}kr\cos\alpha}(\sin\alpha \pm \sin\theta_\pm)S(\alpha \pm \Phi)\mathrm{d}\alpha = 0, \quad 0 < r < +\infty. \tag{50}$$

Rewriting these as in Eqs. (3) and (15), and then invoking the nullification theorem leads to a pair of functional equations

$$(\sin\alpha \pm \sin\theta_\pm)S(\alpha \pm \Phi) - (-\sin\alpha \pm \sin\theta_\pm)S(-\alpha \pm \Phi) = C^\pm \sin\alpha, \tag{51}$$

with arbitrary constants $C^\pm$ on the right-hand sides. Taking the limit $\mathrm{Im}\,\alpha \to \infty$ implies the relations $C^+ = C^- = S(+\mathrm{i}\infty) + S(-\mathrm{i}\infty)$, which, according to the normalisation condition (4), give $C^\pm = 0$. Thus, the functional equations to be solved are

$$(\sin\alpha \pm \sin\theta_\pm)S(\alpha \pm \Phi) - (-\sin\alpha \pm \sin\theta_\pm)S(-\alpha \pm \Phi) = 0. \tag{52}$$

The required solution of this system must be bounded at an infinitely distant point, $\alpha = \infty$, and also satisfy the regularity condition [15]. The latter means that the function

$$S(\alpha) - \frac{U_0}{\alpha - \phi_0} \tag{53}$$

should be regular within the strip $\Pi_0 = \{\alpha : |\mathrm{Re}\,\alpha| \le \Phi\}$, which is necessary to reproduce the incident field (47).

In what follows we adhere to the procedure briefly outlined by Malyuzhinets in 1958 [15]. Suppose that a function $\Psi(\alpha)$ is a particular solution to the functional equation (52), which has no poles and zeros in the strip Π_0. Then, the substitution

$$S(\alpha) = U_0 \frac{\Psi(\alpha)}{\Psi(\phi_0)}\sigma(\alpha) \tag{54}$$

reduces the system (52) to the simple equations

$$\sigma(\alpha \pm \Phi) - \sigma(-\alpha \pm \Phi) = 0. \tag{55}$$

Solutions to these equations can be composed using trigonometric functions, such as $\cos[\nu(\alpha+\Phi)]$ or $\sec[\nu(\alpha+\Phi)]$ with

$$\nu = \frac{\pi}{2\Phi}, \tag{56}$$

and the one satisfying the regularity condition is

$$\sigma(\alpha) = \frac{\nu\cos(\nu\phi_0)}{\sin(\nu\alpha) - \sin(\nu\phi_0)}. \tag{57}$$

The key step in the solution procedure is, therefore, to construct the auxiliary function $\Psi(\alpha)$ which satisfies the same system of equations as $S(\alpha)$, i.e. (52), but not the regularity condition, since $\Psi(\alpha)$ is not allowed to have any poles in the strip $|\mathrm{Re}\,\alpha| \le \Phi$.

We first rewrite the system satisfied by $\Psi(\alpha)$ in the form

$$\frac{\Psi(\alpha \pm \Phi)}{\Psi(-\alpha \pm \Phi)} = \frac{-\sin\alpha \pm \sin\theta_\pm}{\sin\alpha \pm \sin\theta_\pm}. \tag{58}$$

Then, taking the logarithm of both sides of these equations followed by differentiation with respect to α, yields

$$Y(\alpha + \Phi) + Y(-\alpha + \Phi) = Q_+(\alpha), \qquad Y(\alpha - \Phi) + Y(-\alpha - \Phi) = Q_-(\alpha), \tag{59}$$

where

$$Y(\alpha) = \frac{\mathrm{d}}{\mathrm{d}\alpha}\ln\Psi(\alpha), \tag{60}$$

and

$$Q_{\pm}(\alpha) = \frac{\mathrm{d}}{\mathrm{d}\alpha} \ln\left(\frac{-\sin\alpha \pm \sin\theta_{\pm}}{\sin\alpha \pm \sin\theta_{\pm}} \right) = \pm \frac{2\sin\theta_{\pm}\cos\alpha}{\sin^2\alpha - \sin^2\theta_{\pm}}. \tag{61}$$

Notice that the system of equations (59) satisfied by the logarithmic derivative of $\Psi(\alpha)$, i.e. the function $Y(\alpha)$, has constant coefficients, and can, therefore, be solved by applying the Fourier transform.

The functions $Q_{\pm}(\alpha)$ is regular along the imaginary axis in the complex α-plane, as it is seen from Eq. (61). Thus, the transformation (22) can be applied to the system (59)), giving an algebraic problem for the Fourier transform $y(t)$ of the function $Y(\alpha)$:

$$y(t)\mathrm{e}^{-\mathrm{i}t\Phi} + y(-t)\mathrm{e}^{\mathrm{i}t\Phi} = q_{+}(t), \quad y(t)\mathrm{e}^{\mathrm{i}t\Phi} + y(-t)\mathrm{e}^{-\mathrm{i}t\Phi} = q_{-}(t), \tag{62}$$

where

$$q_{\pm}(t) = -\frac{1}{2\pi}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} Q_{\pm}(\alpha)\mathrm{e}^{\mathrm{i}t\alpha}\mathrm{d}\alpha. \tag{63}$$

Solving the system (62) for the variables $y(t)$ and $y(-t)$ yields

$$y(\pm t) = \mp \frac{\mathrm{i}}{2\sin(2t\Phi)}[q_{-}(t)\mathrm{e}^{\pm\mathrm{i}t\Phi} - q_{+}(t)\mathrm{e}^{\mp\mathrm{i}t\Phi}], \tag{64}$$

which define the same function $y(t)$ because $q_{\pm}(t)$, being Fourier transforms of the even functions $Q_{\pm}(\alpha)$, are both even functions of t. The solution of Eq. (59) is therefore

$$Y(\alpha) = \frac{\mathrm{i}}{2}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} [q_{+}(t)\mathrm{e}^{-\mathrm{i}t\Phi} - q_{-}(t)\mathrm{e}^{\mathrm{i}t\Phi}] \frac{\mathrm{e}^{-\mathrm{i}\alpha t}\mathrm{d}t}{\sin(2t\Phi)}, \tag{65}$$

where the integral should be evaluated in the principal value sense at the point $t = 0$.

The solution (65)) can be expressed in terms of the function η_{Φ} introduced in the foregoing. To this end, we first demonstrate that the functions $q_{\pm}(t)$ are

$$q_{\pm}(t) = \pm\mathrm{i}\frac{\cos[t(\pi/2 - \theta_{\pm})]}{\cos(\pi t/2)}. \tag{66}$$

This follows from the integral representation for $q_{\pm}(t)$,

$$q_{\pm}(t) = \mp\frac{\sin\theta_{\pm}}{\pi}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{e}^{\mathrm{i}t\alpha}\cos\alpha}{\sin^2\alpha - \sin^2\theta_{\pm}}\mathrm{d}\alpha, \tag{67}$$

that results from Eqs. (61) and (63). The change of variable $\alpha = -\mathrm{i}\ln\tau$ gives

$$q_{\pm}(t) = \pm\frac{2\mathrm{i}}{\pi}\sin\theta_{\pm} I(t, \theta_{\pm}), \tag{68}$$

with

$$I(t, \theta) = \int_0^{+\infty} \frac{\tau^t(\tau^2 + 1)}{\tau^4 - 2\tau^2\cos(2\theta) + 1}\mathrm{d}\tau. \tag{69}$$

This can be evaluated if we consider the auxiliary integral

$$I_C(t, \theta) = \int_{C(\varepsilon, R)} \frac{\tau^t(\tau^2 + 1)}{\tau^4 - 2\tau^2\cos(2\theta) + 1}\mathrm{d}\tau \tag{70}$$

over a closed smooth contour $C(\varepsilon, R)$ which is supposed to be located entirely on the sheet of the complex τ-plane cut along the positive real axis, where $0 \le \arg\tau \le 2\pi$ (Fig. 4).

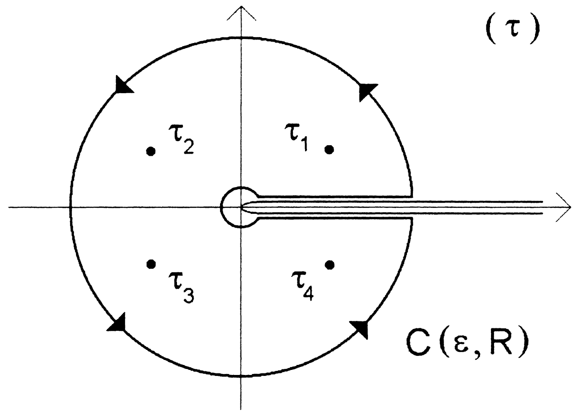

Fig. 4. Integration contour relevant to the construction of $\Psi(\alpha)$.

The contour $C(\varepsilon, R)$ is around the branch cut, bypassing the branch point $\tau = 0$ on a circle of small radius ε, and bypassing the point at infinity along a circle of large radius R so as to enclose the points $\tau_1 = \exp(\mathrm{i}\theta)$, $\tau_2 = \exp[\mathrm{i}(\pi - \theta)]$, $\tau_3 = \exp[\mathrm{i}(\pi + \theta)]$, and $\tau_4 = \exp[\mathrm{i}(2\pi - \theta)]$ at which the integrand in (70) has poles. Assuming that $-1 < \mathrm{Re}\, t < 2$ one may take the limits $\varepsilon \to 0$ and $R \to +\infty$, which leads to the relation

$$I_C(t, \theta) = (1 - \mathrm{e}^{2\pi \mathrm{i} t}) I(t, \theta). \tag{71}$$

On the other hand, the integral (70) can be evaluated by applying Cauchy's theorem, as the sum of residues at the poles of the integrand enclosed by the integration contour:

$$I_C(t, \theta) = \frac{\pi \mathrm{i}}{2} \sum_{n=1}^{4} \frac{\tau_n^{t-1}(\tau_n^2 + 1)}{\tau_n^2 - \cos(2\theta)}, \tag{72}$$

which because of Eqs. (68) and (71) gives (66), as claimed.

Replacing the cosine functions in Eq. (66) with

$$\cos\left[t\left(\frac{\pi}{2} - \theta_\pm\right)\right] = \frac{1}{2}[\mathrm{e}^{\mathrm{i}t(\pi/2 - \theta_\pm)} + \mathrm{e}^{-\mathrm{i}t(\pi/2 - \theta_\pm)}]$$

followed by their insertion into Eq. (65) yields

$$\begin{aligned} Y(\alpha) = {} & \eta_\Phi\left(\alpha + \Phi + \frac{\pi}{2} - \theta_+\right) + \eta_\Phi\left(\alpha + \Phi - \frac{\pi}{2} + \theta_+\right) \\ & + \eta_\Phi\left(\alpha - \Phi + \frac{\pi}{2} - \theta_-\right) + \eta_\Phi\left(\alpha - \Phi - \frac{\pi}{2} + \theta_-\right), \end{aligned} \tag{73}$$

where we have used the fundamental integral representation (24)) for the η_Φ function.

The function $\Psi(\alpha)$ is related to its logarithmic derivative $Y(\alpha)$ by the formula

$$\Psi(\alpha) = \exp\left(\int_0^\alpha Y(t)\, \mathrm{d}t\right), \tag{74}$$

which, because of (73) and the definition (34) of the special function ψ_Φ, gives

$$\begin{aligned} \Psi(\alpha) = {} & \psi_\Phi\left(\alpha + \Phi + \frac{\pi}{2} - \theta_+\right) \psi_\Phi\left(\alpha + \Phi - \frac{\pi}{2} + \theta_+\right) \times \psi_\Phi\left(\alpha - \Phi - \frac{\pi}{2} + \theta_-\right) \\ & \times \psi_\Phi\left(\alpha - \Phi + \frac{\pi}{2} - \theta_-\right). \end{aligned} \tag{75}$$

One may directly verify the validity of this formula by showing that it does satisfy the functional equations (58). Substituting $\Psi(\alpha)$ from Eq. (75) directly into Eq. (58), and using the fact that the Malyuzhinets function is an even function of its argument, leads to the relations

$$\frac{\Psi(\alpha\pm\Phi)}{\Psi(-\alpha\pm\Phi)}=\frac{\psi_\Phi(\alpha\pm2\Phi+\pi/2-\theta_\pm)\psi_\Phi(\alpha\pm2\Phi-\pi/2+\theta_\pm)}{\psi_\Phi(\alpha\mp2\Phi-\pi/2+\theta_\pm)\psi_\Phi(\alpha\mp2\Phi+\pi/2-\theta_\pm)}. \tag{76}$$

These reduce to Eq. (58) by applying the functional property (36), thus verifying the fundamental identity (75).

The poles and zeros of $\Psi(\alpha)$ are simply those of the four Malyuzhinets functions appearing in Eq. (75). Correspondingly, the poles of $\Psi(\alpha)$ belong to the four families of points:

$$\alpha=\Phi+\pi+\theta_++4n\Phi,\ -3\Phi-\pi-\theta_+-4n\Phi,\ -\Phi-\pi-\theta_--4n\Phi,\ 3\Phi+\pi+\theta_-+4n\Phi. \tag{77}$$

with $n=0,1,2,\ldots$. Note that the absence of poles and zeros of $\Psi(\alpha)$ in the strip Π_0, which guarantees that the transform function $S(\alpha)$ given by Eq. (54) satisfies the regularity condition (53), is a consequence of the similar analytic behaviour of $\psi_\Phi(\alpha)$ for $|\mathrm{Re}\,\alpha|<2\Phi+\frac{1}{2}\pi$.

The asymptotic behaviour of the function $\Psi(\alpha)$ as $\mathrm{Im}\,\alpha\to\pm\infty$ is given by

$$\Psi(\alpha)=\frac{1}{4}\psi_\Phi^4\left(\frac{\pi}{2}\right)\mathrm{e}^{\mp\mathrm{i}\nu\alpha}[1+o(1)], \tag{78}$$

which results from (75) and the property (44). Eqs. (54) and (57), combined with the relation (78), ensure the boundedness of the spectral function $S(\alpha)$ at the infinitely distant point $\mathrm{Im}\,\alpha=\infty$. This, along with the identity (5), specifies the finite value of the potential function $u(r,\phi)$ at the edge of the wedge as

$$\lim_{r\to0}u(r,\phi)=\nu U_0\psi_\Phi^4\left(\frac{\pi}{2}\right)\frac{\cos(\nu\phi_0)}{\Psi(\phi_0)}. \tag{79}$$

This completes the construction of the auxiliary function $\Psi(\alpha)$ and, therefore, the proper transform function $S(\alpha)$ relevant to diffraction from an impedance wedge, which is given by (54) together with (57) and (75). Before proceeding further it is useful to discuss some limiting cases.

Using the functional relation (37), one may rewrite Eq. (75) as follows

$$\begin{aligned}\Psi(\alpha)=\psi_\Phi^4\left(\frac{\pi}{2}\right)\cos\left[\frac{\pi}{4\Phi}(\alpha+\Phi-\theta_+)\right]\cos\left[\frac{\pi}{4\Phi}(\alpha-\Phi+\theta_-)\right]\\ \times\frac{\psi_\Phi(\alpha+\Phi-\frac{\pi}{2}+\theta_+)\psi_\Phi(\alpha-\Phi+\frac{\pi}{2}-\theta_-)}{\psi_\Phi(\alpha+\Phi-\frac{\pi}{2}-\theta_+)\psi_\Phi(\alpha-\Phi+\frac{\pi}{2}+\theta_-)}.\end{aligned} \tag{80}$$

Thus, when the wedge faces are identical, i.e. $\theta_+=\theta_-$, the function $\Psi(\alpha)$ is an even function of its argument.

Furthermore, under certain circumstances, $\Psi(\alpha)$ and $S(\alpha)$ may become 2π-periodic functions of α. More precisely, this is the case for arbitrary face impedances if $\Phi=\pi/(4m)$ with m integer, while for equal impedances of the faces the property holds when $\Phi=\pi/(2q)$ and $q=1,2,3,\ldots$. All these particular cases relate to the case of an interior wedge, and, as one may prove by a simple analysis of the Sommerfeld integral (2) assuming its integrand function to be 2π-periodic, the solution of the diffraction problem $u(r,\phi)$ can be then expressed without integration, through a finite number of residue contributions [15].

To prove the periodicity, first observe that the function $\sigma(\alpha)$ is necessarily 2π-periodic when $\Phi=\pi/(2q)$ with q an arbitrary integer, which results immediately from its representation (57). Next consider the auxiliary function $\Psi(\alpha)$ given by (75) and notice the relation

$$\Psi(\alpha+2\pi)=\Psi(\alpha)\prod_{k=1}^{4}\frac{\cos[\frac{\pi}{4\Phi}(\alpha_k+\frac{3}{2}\pi)]}{\cos[\frac{\pi}{4\Phi}(\alpha_k+\frac{1}{2}\pi)]} \tag{81}$$

with $\alpha_1 = \alpha + \Phi - \frac{1}{2}\pi + \theta_+$, $\alpha_2 = \alpha + \Phi + \frac{1}{2}\pi - \theta_+$, $\alpha_3 = \alpha - \Phi - \frac{1}{2}\pi + \theta_-$, and $\alpha_4 = \alpha - \Phi + \frac{1}{2}\pi - \theta_-$, which follows from repeated use of the functional property (37). The cosine functions in the numerator of Eq. (81) can be represented as

$$\cos\left[\frac{\pi}{4\Phi}\left(\alpha_k + \frac{3}{2}\pi\right)\right] = \cos\left[\frac{\pi}{4\Phi}\left(\alpha_k + \frac{1}{2}\pi\right)\right]\cos\left(\frac{\pi^2}{4\Phi}\right) - \sin\left[\frac{\pi}{4\Phi}\left(\alpha_k + \frac{1}{2}\pi\right)\right]\sin\left(\frac{\pi^2}{4\Phi}\right),$$

which simplifies to $(-1)^m \cos[\frac{\pi}{4\Phi}(\alpha_k + \frac{\pi}{2})]$ when $\Phi = \pi/(4m)$ with m integer. Thus, in the latter case the product in Eq. (81) becomes unity, implying the 2π-periodicity of the $\Psi(\alpha)$ function.

If the impedances of the faces are the same, that is, $\theta_+ = \theta_- = \theta$, this periodicity property of $\Psi(\alpha)$ occurs for the wider set of vertex angles given by $\Phi = \pi/(2q)$ with $q = 1, 2, 3, \ldots$, which for even q reproduces the previous case of arbitrary impedances. To make this clear, we rewrite the products of cosine functions from (81) as

$$\prod_{k=1}^{4}\cos\left[\frac{\pi}{4\Phi}\left(\alpha_k + \frac{1}{2}\pi\right)\right] = \frac{1}{4}\cos\left[\frac{\pi}{2\Phi}(\alpha + \theta)\right]\cos\left[\frac{\pi}{2\Phi}(\pi + \alpha - \theta)\right],$$

$$\prod_{k=1}^{4}\cos\left[\frac{\pi}{4\Phi}\left(\alpha_k + \frac{3}{2}\pi\right)\right] = \frac{1}{4}\cos\left[\frac{\pi}{2\Phi}(\pi + \alpha + \theta)\right]\cos\left[\frac{\pi}{2\Phi}(2\pi + \alpha - \theta)\right],$$

yielding

$$\Psi(\alpha + 2\pi) = \Psi(\alpha)\prod_{k=1}^{2}\frac{\cos[\frac{\pi}{2\Phi}(\beta_k + \pi)]}{\cos(\pi\beta_k/(2\Phi))} \tag{82}$$

with $\beta_1 = \alpha + \theta$ and $\beta_2 = \alpha + \pi - \theta$. Again, one may readily check that the product in Eq. (82) is always unity if $\Phi = \pi/(2q)$ with integer q. Thus, we have established our claim concerning 2π-periodicity of the transform function $S(\alpha)$.

Now consider what happens to the solution (54) in the particular cases of ideal boundaries. For vanishing values of the Brewster angles, $|\theta_\pm| \to 0$, Eq. (80) reduces to

$$\Psi(\alpha) \to \frac{1}{2}\psi_\Phi^4\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi\alpha}{2\Phi}\right), \tag{83}$$

which, together with Eq. (54), yields

$$S(\alpha) \to U_0 \frac{\nu\cos(\nu\alpha)}{\sin(\nu\alpha) - \sin(\nu\phi_0)}. \tag{84}$$

Eq. (84) is a well-known expression for the transform function of a wedge with Neumann boundary conditions [1,2].

For a wedge with one face acoustically hard, say that at $\phi = -\Phi$, while the other face has finite, non-zero impedance, we get

$$S(\alpha) = U_0\,\sigma(\alpha)\frac{\tilde{\Psi}(\alpha)}{\tilde{\Psi}(\phi_0)}\frac{\cos[\frac{\pi}{4\Phi}(\alpha - \Phi)]}{\cos[\frac{\pi}{4\Phi}(\phi_0 - \Phi)]}, \tag{85}$$

with

$$\tilde{\Psi}(\alpha) = \psi_\Phi\left(\alpha + \Phi + \frac{\pi}{2} - \theta_+\right)\psi_\Phi\left(\alpha + \Phi - \frac{\pi}{2} + \theta_+\right). \tag{86}$$

The limit of Dirichlet boundary conditions, i.e. $\mathrm{Im}|\theta_\pm| \to \infty$, can be treated on the basis of the asymptotic estimate (44). This gives

$$|\Psi(\alpha)| \to \frac{1}{4}\psi_\Phi^4\left(\frac{\pi}{2}\right)\exp\left[\frac{\pi}{4\Phi}(|\mathrm{Im}\,\theta_+| + |\mathrm{Im}\,\theta_-|)\right] \tag{87}$$

for the Ψ function, and therefore, the quotient $\Psi(\alpha)/\Psi(\phi_0)$ in Eq. (54) tends to unity, yielding another well-known limiting expression

$$S(\alpha) \to U_0 \frac{\nu \cos(\nu\phi_0)}{\sin(\nu\alpha) - \sin(\nu\phi_0)}, \tag{88}$$

relevant to acoustically soft boundaries. If only one face of the wedge is acoustically soft, for instance $\phi = -\Phi$, then taking the limit $|\mathrm{Im}\,\theta_-| \to \infty$ in Eq. (54) gives

$$S(\alpha) = U_0 \frac{\tilde{\Psi}(\alpha)}{\tilde{\Psi}(\phi_0)} \sigma(\alpha) \tag{89}$$

with $\tilde{\Psi}(\alpha)$ defined in (86).

We conclude this section by mentioning that the transform $S(\alpha)$ from Eq. (54) is clearly a meromorphic function of the complex variable α. Its singularities are poles located at two subsets of points, one of which, given by Eq. (77), is related to the auxiliary function $\Psi(\alpha)$, whereas the other subset comes from the trigonometric function $\sigma(\alpha)$ and these latter are as follows:

$$\alpha = (-1)^m \phi_0 + 2m\Phi, \quad m = 0, \pm 1, \pm 2, \ldots \tag{90}$$

Consequently, the transform $S(\alpha + \phi)$ in Eq. (2) is analytic inside the loops $\gamma_\pm$ as long as they do not enclose any poles of the functions $\Psi(\alpha + \phi)$ and $\sigma(\alpha + \phi)$. This is essentially the definition of the loops, which should reside in the region where $|\mathrm{Im}\,\alpha| > V$ with

$$V = \sup(|\mathrm{Im}\,\theta_\pm|, |\mathrm{Im}\,\phi_0|). \tag{91}$$

4. Near-field approximations

4.1. Series representation of the Malyuzhinets solution

The solution in Section 3 is in the form of a Sommerfeld integral (2) with its transform function given by Eq. (54). This is well suited to analysing the wave function $u(r, \phi)$ far from the edge, where $kr \gg 1$ and the saddle point technique is applicable. On the other hand, this technique cannot be applied to evaluate the Sommerfeld integral in the near and intermediate zones where $kr \le 1$. Another representation of the Malyuzhinets solution is therefore required to describe the wave field diffracted by an impedance wedge in the vicinity of its edge. In this section we represent the Malyuzhinets solution as a series of Bessel functions [38], which in the limits of Dirichlet–Neumann boundary conditions reduces to the well-known expressions [1–7]

$$u(r, \phi) = 4\nu U_0 \sum_{p=0}^{\infty} \delta_p \mathrm{e}^{-\mathrm{i}\nu_p \pi/2} J_{\nu_p}(kr) \chi[\nu_p(\Phi + \phi)] \chi[\nu_p(\Phi + \phi_0)], \tag{92}$$

where $J_{\nu_p}(kr)$ are the Bessel functions of the first kind, $\nu = \pi/(2\Phi)$ is defined in (56), and $\delta_p = 1$ if $p \ge 1$. The values of the other parameters δ_0, ν_p as well as the definition of $\chi(\tau)$ depend upon the type of boundary conditions. They are as follows: $\delta_0 = \frac{1}{2}$, $\nu_p = \nu p$, and $\chi(\tau) = \cos\tau$ if $\theta_\pm = 0$ (hard faces); $\delta_0 = 1$, $\nu_p = \nu p$, and $\chi(\tau) = \sin\tau$ if $\mathrm{Im}\,\theta_\pm = \infty$ (soft faces); $\delta_0 = 1$, $\nu_p = \nu(p + \frac{1}{2})$, and $\chi(\tau) = \sin\tau$ if $\theta_+ = 0$ and $\mathrm{Im}\,\theta_- = \infty$ (mixed boundary conditions).

We begin by noting that the Sommerfeld integral in (3) involves integration over one loop γ_+ extending to $\mathrm{Im}\,\alpha \to +\infty$ and residing entirely in the upper half-plane $\mathrm{Im}\,\alpha > 0$. By referring to the analytical form of the expansion of the trigonometrical factor in Eq. (54),

$$\sigma(\pm\alpha + \phi) = \mp 2\nu \sum_{n=1}^{\infty} a_{n-1}(\pm\phi_0) \exp\left[\mathrm{i}n\nu(\alpha \pm \varphi) + \mathrm{i}n\frac{\pi}{2}\right], \tag{93}$$

with $a_{n-1}(\phi_0) = \sin[n(\nu\varphi_0+\pi/2)]$, one may search for the expansion of $S(\pm\alpha+\phi)$ into a sequence of exponential functions $\{\exp(\mathrm{i}\nu_p\alpha)\}_{p=0}^{+\infty}$ such that $0 \le \mathrm{Re}\,\nu_0 < \mathrm{Re}\,\nu_1 < \cdots$

This is clearly achieved if the remaining factor in Eq. (54), that is, $\Psi(\pm\alpha+\phi)$ is expanded in a similar form. To this end, it is useful to exploit series representations (42) from Section 2.3. Since $\mathrm{Im}\,\alpha > V$ throughout the whole contour γ_+ we may replace all the Malyuzhinets functions occurring in (75) with the series representations (42), which yields

$$\Psi(\pm\alpha+\phi) = \frac{1}{4}\psi_\Phi^4\left(\frac{\pi}{2}\right)\mathrm{e}^{-\mathrm{i}\nu(\alpha\pm\phi)}\exp\left[\sum_{k=1}^{\infty} b_k^{\pm}\mathrm{e}^{\mathrm{i}\nu k(\alpha\pm\phi)} + \sum_{k=1}^{\infty} c_k^{\pm}\mathrm{e}^{\mathrm{i}(2k-1)(\alpha\pm\phi)}\right], \tag{94}$$

where

$$\begin{aligned} b_k^{\pm} &= -\frac{\cos[\nu k(\pi/2-\theta_+)]}{k\cos(\pi\nu k/2)}\mathrm{e}^{\mp\mathrm{i}\pi k/2} - \frac{\cos[\nu k(\pi/2-\theta_-)]}{k\cos(\pi\nu k/2)}\mathrm{e}^{\pm\mathrm{i}\pi k/2}, \\ c_k^{\pm} &= \frac{\sin[(2k-1)\theta_+]\mathrm{e}^{\pm\mathrm{i}(2k-1)\Phi}}{(k-\frac{1}{2})\sin[2\Phi(2k-1)]} + \frac{\sin[(2k-1)\theta_-]\mathrm{e}^{\mp\mathrm{i}(2k-1)\Phi}}{(k-\frac{1}{2})\sin[2\Phi(2k-1)]}. \end{aligned} \tag{95}$$

The expansions (93) and (94) can now be used to rewrite the transform function as a double series

$$S(\alpha) = \sum_{p,q=0}^{+\infty} S_{pq}^{\pm}\mathrm{e}^{\pm\mathrm{i}\nu_{pq}\alpha}, \tag{96}$$

where $S_{pq}^{\pm}$ are constant coefficients and $\nu_{pq} = \nu p + q$. Inserting Eq. (96) in Eq. (3) and using the integral representation of the Bessel function

$$J_\nu(z) = -\frac{1}{2\pi}\mathrm{e}^{\mathrm{i}\nu\pi/2}\int_{\gamma_+}\mathrm{e}^{-\mathrm{i}z\cos\alpha+\mathrm{i}\nu\alpha}\mathrm{d}\alpha, \tag{97}$$

converts the Malyuzhinets solution into the series expression of the form

$$u(r,\phi) = U_0\frac{\nu\psi_\Phi^4(\pi/2)}{2\Psi(\phi_0)}\sum_{p,q=0}^{\infty} J_{\nu_{pq}}(kr)\mathrm{e}^{-\mathrm{i}\nu_{pq}\pi/2} f_p(\phi_0)(g_q^+\mathrm{e}^{\mathrm{i}p\pi/2+\mathrm{i}\nu_{pq}\phi} + g_q^-\mathrm{e}^{-\mathrm{i}p\pi/2-\mathrm{i}\nu_{pq}\phi}). \tag{98}$$

The coefficients $f_p(\phi_0)$ and $g_q^{\pm}$ may be obtained from the generating functions

$$\sum_{q=0}^{\infty} g_q^{\pm} z^q = \exp\left(\sum_{k=1}^{\infty} c_k^{\pm} z^{2k-1}\right), \tag{99}$$

$$\sum_{p=0}^{\infty} f_p(\phi_0) z^p = \sum_{n=0}^{\infty} a_n(\phi_0) z^n \exp\left(\sum_{k=1}^{\infty} b_k z^k\right), \tag{100}$$

where $b_k = b_k^- e^{i\pi k/2}$, while $a_n(\phi_0)$, b_k^-, and $c_k^{\pm}$ are defined in Eqs. (93) and (95), respectively.

By explicitly expanding the exponential functions in (99) and (100) into power series of z, the rules (99) and (100) can be transformed into recurrent sequences convenient for numerical computations [38]. Specifically, the coefficients $f_p(\phi_0)$ are given by the relations

$$f_p(\phi_0) = \sum_{k=0}^{p} a_{p-k}(\phi_0)\, d_k, \tag{101}$$

where

$$d_0 = 1, \qquad d_k = \sum_{m=1}^{k} \frac{1}{m!} \mathrm{d}_{k-m}^{(m)} \quad k \geq 1,$$

and

$$d_0^{(j)} = b_1^j, \qquad d_m^{(j)} = \frac{1}{mb_1} \sum_{n=1}^{m} b_{n+1} \mathrm{d}_{m-n}^{(j)} (nj - m + n).$$

The first three coefficients in Eq. (101) are as follows: $f_0(\phi_0) = a_0(\phi_0)$, $f_1(\phi_0) = a_1(\phi_0) + b_1 a_0(\phi_0)$, and $f_2(\phi_0) = a_2(\phi_0) + b_1 a_1(\phi_0) + a_0(\phi_0)(b_2 + b_1^2/2)$.

The coefficients $g_q^{\pm}$ with $q = 1, 2, \ldots$ result from the successive use of the matrix recurrent relations

$$\sin[2\Phi(q+1)] \boldsymbol{G}_{q+1} = K_q \boldsymbol{G}_q + L_q \boldsymbol{G}_{q-1}, \tag{102}$$

with

$$K_q = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\Phi(2q+1)} \sin\theta_+ + \mathrm{e}^{-\mathrm{i}\Phi(2q+1)} \sin\theta_- & \mathrm{e}^{\mathrm{i}\Phi} \sin\theta_+ + \mathrm{e}^{-\mathrm{i}\Phi} \sin\theta_- \\ \mathrm{e}^{-\mathrm{i}\Phi} \sin\theta_+ + \mathrm{e}^{\mathrm{i}\Phi} \sin\theta_- & \mathrm{e}^{-\mathrm{i}\Phi(2q+1)} \sin\theta_+ + \mathrm{e}^{\mathrm{i}\Phi(2q+1)} \sin\theta_- \end{pmatrix},$$

and

$$L_q = \begin{pmatrix} \sin(2q\Phi) & -\sin(2\Phi) \\ -\sin(2\Phi) & \sin(2q\Phi) \end{pmatrix}.$$

Here $\boldsymbol{G}_q = (g_q^+, g_q^-)^{\mathrm{T}}$ and the initial values for the recurrence procedure (102) are $\boldsymbol{G}_{-1} = (0, 0)^{\mathrm{T}}$ and $\boldsymbol{G}_0 = (1, 1)^{\mathrm{T}}$. The first three coefficients are found from (102) as follows: $g_0^{\pm} = 1$, $g_1^{\pm} = c_1^{\pm}$, and $g_2^{\pm} = (c_1^{\pm})^2/2$.

In the limit as $\theta_{\pm} \to 0$ the coefficients $g_q^{\pm}$ and $f_p(\phi_0)$ become $g_0^{\pm} = 1$ and $g_q^{\pm} = 0$ with $q \geq 1$ and $f_0(\phi_0) = a_0(\phi)$, $f_1(\phi_0) = a_1(\phi)$, and $f_p(\phi_0) = a_p(\phi) - a_{p-2}(\phi_0)$ with $p \geq 2$, respectively. This reduces the double series (98) to the single one (92) for the case of Neumann boundary conditions if the limiting expression (83) for the function $\Psi(\phi_0)$ is accounted for.

To consider the alternative case of the Dirichlet boundary conditions obtained in the limit $|\mathrm{Im}\,\theta_{\pm}| \to \infty$ we need to return to the Sommerfeld integral (3) and take the limit with respect to the Brewster angles in the transform functions $S(\pm\alpha + \phi)$. If both faces of the wedge are acoustically soft, the limiting expression given by (88) involves only trigonometric functions and the corresponding Sommerfeld integral can be readily transformed into the Bessel function series (92) (see, for example, [3]).

In order to obtain a series representation for the case of a wedge with one face acoustically soft and the other of finite impedance, one should start with the formula (89) and expand it into a sequence of exponential functions. The expansion is then inserted into (3) and the integral representation (97) for Bessel functions is again used, yielding the series

$$u(r, \phi) = 2\nu U_0 \frac{\psi_\Phi^2(\pi/2)}{\tilde{\Psi}(\phi_0)} \sum_{p,q=0}^{+\infty} J_{\tilde{\nu}_{pq}}(kr) \mathrm{e}^{-\mathrm{i}\tilde{\nu}_{pq}\pi/2} \tilde{f}_p(\phi_0) \tilde{g}_q \sin[\tilde{\nu}_{pq}(\Phi + \phi)], \tag{103}$$

where $\tilde{\nu}_{pq} = \nu(p + \frac{1}{2}) + q$, and the function $\tilde{\Psi}(\phi_0)$ is defined by Eq. (87).

The coefficients $\tilde{f}_p(\phi_0)$ and $\tilde{g}_q$ can be found by expanding the generating functions similar to those in Eqs. (99) and (100) with

$$\tilde{b}_k = (-1)^{k+1} \frac{\cos[\nu k(\pi/2 - \theta_+)]}{k \cos(\pi\nu k/2)}, \qquad \tilde{c}_k^+ = \frac{\sin[(2k-1)\theta_+]}{(k - \frac{1}{2}) \sin[2\Phi(2k-1)]},$$

in place of b_k and $c_k^{\pm}$, respectively. The corresponding recurrence procedure for $\tilde{f}_p(\phi_0)$ results from Eq. (101) on replacing b_k with $\tilde{b}_k$, whereas the one for $\tilde{g}_q$ is

$$\tilde{g}_{q+1} = 2\sin\theta_+ \frac{\cos(2\Phi q)}{\sin[2\Phi(q+1)]}\tilde{g}_q + \frac{\sin[2\Phi(q-1)]}{\sin[2\Phi(q+1)]}\tilde{g}_{q-1}, \tag{104}$$

with $\tilde{g}_{-1} = 0$, $\tilde{g}_0 = 1$, and $q = 0, 1, 2, \ldots$ One may check that in the limit $\theta_+ \to 0$ the series (103) reduces to (92) for the case of mixed boundary conditions with $\theta_+ = 0$ and $\mathrm{Im}\,\theta_- = \infty$.

The series (98) and (103) converge absolutely. This can be seen by noting that for $n \to \infty$ the recurrence relations (101), (102) and (104) lead to the estimates

$$g_n^{\pm} = \mathrm{O}(\mathrm{e}^{nV}), \qquad f_n(\phi_0) = \mathrm{O}(\mathrm{e}^{\nu nV}), \qquad \tilde{g}_n = \mathrm{O}(\mathrm{e}^{n|\mathrm{Im}\,\theta_+|}), \qquad \tilde{f}_n(\phi_0) = \mathrm{O}(\mathrm{e}^{\nu n|\mathrm{Im}\,\theta_+|}), \tag{105}$$

with V defined in Eq. (91). Since for large and positive ν the Bessel functions $J_\nu(kr)$ behave like $\mathrm{O}[(kr/\nu)^\nu]$, the estimates (105) guarantee the convergence of the series.

The members of (98) and (103) become negligible when $\nu_{pq} > \mathrm{O}(kr\mathrm{e}^V)$ and $\tilde{\nu}_{pq} > \mathrm{O}(kr\mathrm{e}^{|\mathrm{Im}\,\theta_+|})$, respectively. This means that the series representations are particularly useful for small and moderate values of kr, since retaining a few leading terms in Eqs. (98) and (103) may provide accurate approximations for the whole series.

We may now consider the behaviour of the Malyuzhinets solution near the edge of the wedge, when $kr \to 0$. Replacing the Bessel functions in Eq. (98) with their ascending series

$$J_\nu(kr) = \left(\frac{kr}{2}\right)^\nu \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\Gamma(\nu+j+1)} \left(\frac{kr}{2}\right)^{2j}, \tag{106}$$

where $\Gamma(\alpha)$ is the Gamma function, transforms the solution into a power series of the form

$$u(r,\phi) = \sum_{p=0}^{\infty} (kr)^{\nu p} \sum_{m=0}^{\infty} A_{pm}(\phi)(kr)^m, \tag{107}$$

with coefficients directly related to $g_q^{\pm}$ and $f_p(\phi_0)$. If kr is sufficiently small, we may retain only a few leading terms in (107), yielding the approximation

$$u = A_{00} + A_{10}(\phi)(kr)^\nu + A_{01}(\phi)kr + \mathrm{O}[(kr)^{2\delta}], \tag{108}$$

where $\delta = \inf(\nu, 1)$ and

$$A_{00} = \nu U_0 \frac{\psi_\Phi^4(\pi/2)}{\Psi(\phi_0)} \cos(\nu\phi_0),$$

$$A_{01}(\phi) = -\mathrm{i}\nu U_0 \frac{\psi_\Phi^4(\pi/2)\cos(\nu\phi_0)}{\Psi(\phi_0)\sin(2\Phi)}[\sin\theta_+ \cos(\phi+\Phi) + \sin\theta_- \cos(\phi-\Phi)],$$

$$A_{10}(\phi) = U_0 \frac{\nu\psi_\Phi^4(\pi/2)\sin(\nu\phi)\cos(\nu\phi_0)}{(2\mathrm{i})^\nu\Gamma(\nu+1)\Psi(\phi_0)}[2\sin(\nu\phi_0) - b_1], \tag{109}$$

with $b_1 = \mathrm{i}b_1^-$ defined from Eq. (95).

Eq. (108) describes the behaviour of the wave field near the edge of an impedance wedge. A_{00} is the main term. It characterises the value of the wave potential $u(r,\phi)$ at the edge, and agrees with the limiting expression (79), as expected. The next two terms of Eq. (108) are necessary to describe the edge behaviour of the derivatives $\partial u/\partial r$ and $\partial u/(r\partial\phi)$ that represent the radial and angular components of the particle speed in the acoustic case or the associated components of the electric/magnetic fields in the electromagnetic case. If the wedge is acute, that is, $\Phi > \pi/2$, then $\nu < 1$ and the first derivatives of the function $u(r,\phi)$ with respect to any coordinate have a singularity of the form

$A_{10}(\phi)(kr)^{\nu-1}$ at the edge, which arises from the second term in Eq. (108). Alternatively, in the case of an internal wedge, i.e. $\Phi < \pi/2$, the first derivatives are bounded at the edge and their behaviour depends on the third member in the right part of Eq. (108), proportional to $A_{01}(\phi)$.

The expansion (108) is also applicable if one or both faces of the wedge is acoustically hard, which can be considered by taking the appropriate limits with respect to the impedance parameters in equations (109). In the case that one of the faces is acoustically soft, corresponding to the Dirichlet boundary condition, the formula (108) is replaced by the following:

$$u(r,\phi) = B_{00}(\phi)(kr)^{\nu/2} + \mathrm{O}[(kr)^{\delta+\nu/2}], \tag{110}$$

with

$$B_{00}(\phi) = U_0 \frac{2\nu\psi_\Phi^2(\pi/2)\sin[\nu(\phi_0+\Phi)]}{(2\mathrm{i})^{\nu/2}\Gamma(\nu/2+1)\tilde{\Psi}(\phi_0)} \sin\left[\frac{\nu}{2}(\Phi+\varphi)\right],$$

which follows from the relevant series representation (103).

By contrast with Eq. (108), the expression (110) implies that irrespective of the wedge angle Φ the wave potential $u(r,\phi)$ vanishes as $kr \to 0$ while its first derivatives $\partial u/\partial r$ and $\partial u/(r\partial\phi)$ tend to zero if $\Phi < \pi/4$ or are singular at the edge if $\Phi > \pi/4$, in accordance with the estimate $u = \mathrm{O}[(kr)^{-1+\nu/2}]$. Notice that this singularity of the field derivatives is stronger than that for the wedge with faces acoustically hard or of non-zero impedance.

As it was pointed out in Section 2.3, certain terms in the expansion given by (42) and (43) may become infinite when $\Phi = \Phi_{nm}$, where $\Phi_{nm} = \pi n/[2(2m-1)]$ with n and m integers. However, such singularities can be shown to cancel each other so that their sum (43) remains bounded, which means that these cases can be treated by simply taking the limit $\Phi \to \Phi_{nm}$ in the series expressions presented so far in this section. As a result of cancelling the singularities the derivatives of the Bessel function $J_\nu(kr)$ with respect to ν may enter the series (98) and (103). Correspondingly, the logarithmic terms of the forms $(kr)^p \ln^q(kr)$ with integer p and q occur in the expansions (107). The analytic structure of the resulting series depends on the particular values of m and n in Φ_{nm}. Two specific examples are presented next, for $n, m = 1$ (a flat surface with an impedance step), and for $n = 2, m = 1$ (an impedance half-plane).

Taking the limit $\Phi \to \pi/2$ in Eq. (107) gives an expansion of the form

$$u(r,\phi) = \sum_{p=0}^{\infty}(kr)^p \sum_{m=0}^{p} \tilde{A}_{pm}(\phi)\ln^m(kr), \tag{111}$$

which, for sufficiently small values of kr, can be replaced by the formula

$$u(r,\phi) = \tilde{A}_{00} + \tilde{A}_{11}(\phi)kr\ln(kr) + \mathrm{O}(kr).$$

Here

$$\tilde{A}_{11}(\phi) = U_0 \frac{\psi_\Phi^4(\pi/2)}{\mathrm{i}\pi\Psi(\phi_0)}(\sin\theta_+ - \sin\theta_-)\cos\phi_0\sin\phi,$$

while the coefficient $\tilde{A}_{00}$ equals A_{00} from Eq. (108) with $\Phi = \pi/2$. A consequence of Eq. (111) is that the first derivatives $\partial u/\partial r$ and $\partial u/(r\partial\phi)$ of the wave function for diffraction from an impedance step exhibit a logarithmic singularity at the discontinuity point, rather than an algebraic power law singularity.

In the case of an impedance half-plane, one has the expansion

$$u(r,\phi) = \sum_{p=0}^{\infty}(kr)^p \sum_{m=0}^{\infty} \ln^m(kr)[\tilde{A}_{2p,m}(\phi) + (kr)^{1/2}\tilde{A}_{2p+1,m}(\phi)], \tag{112}$$

with coefficients defined such that $\tilde{A}_{nm}(\phi) = 0$ if $n < m$. Two leading terms of Eq. (112) are:

$$u(r, \phi) = \tilde{A}_{00} + \sqrt{kr}\,\tilde{A}_{10}(\phi) + \mathrm{O}(kr \ln kr),$$

where $\tilde{A}_{00} = \lim_{\Phi\to\pi} A_{00}$ and $\tilde{A}_{10}(\phi) = \lim_{\Phi\to\pi} A_{10}(\phi)$. Thus, at the edge of an impedance half-plane the singular components of the wave field behave as $\mathrm{O}[(kr)^{-1/2}]$.

Similar expansions occur if one face of the half-plane or the impedance step plane is acoustically soft. The expansions differ in their analytical form from (111) and (112) only by a factor $(kr)^{\nu/2}$, and the coefficients in their leading terms are given by $B_{00}(\phi)$ from Eq. (110) with $\Phi \to \pi/2$ or $\Phi \to \pi$, respectively.

4.2. The edge field

The value of the total field at the edge of an impedance wedge is not, in general, an easy quantity to compute because of its dependence on Malyuzhinets functions. Although these are now well documented, and fast algorithms exist for their computation (see Section 2.3), they are still cumbersome to handle as compared with trigonometric functions. In this section we show how the edge field can be described in a simple way, using only trigonometric functions. Also, the edge field is analysed as a function of the face impedances and the vertex angle of the wedge.

We will work with the normalised edge field $u_0(\phi_0) \equiv u(0, \phi)/U_0$, which is the total field at the edge for a plane wave of unit amplitude incident from direction ϕ_0. It follows from the alternative form of $\Psi(\alpha)$ in Eq. (80) that the limiting expression (79) for the edge value of the total field can be rewritten

$$u_0(\phi) = \frac{\nu \cos(\nu\phi) X_+(\phi, \theta_+) X_-(\phi, \theta_-)}{\cos[\frac{\nu}{2}(\phi + \Phi - \theta_+)] \cos[\frac{\nu}{2}(\phi - \Phi + \theta_-)]}, \tag{113}$$

where

$$X_\pm(\phi, \theta) = \frac{\psi_\Phi[\phi \pm (\Phi - \pi/2 - \theta)]}{\psi_\Phi[\phi \pm (\Phi - \pi/2 + \theta)]}. \tag{114}$$

If the impedance of a face is a pure reactance, that is, it is purely imaginary, then the associated angle, θ_+ or θ_-, is also purely imaginary. Let us assume for the moment that this is the case for both faces, then the fact that $\psi_\Phi(\bar{\alpha}) = \overline{\psi_\Phi(\alpha)}$ implies that $|X_\pm(\phi, \theta_\pm)| = 1$. The denominator in Eq. (113) can be further simplified by splitting the terms $\sin[\nu(\phi - \theta_+)]$ and $\sin[\nu(\phi + \theta_-)]$ into their real and imaginary parts, yielding

$$|u_0(\phi)| = \frac{2\nu \cos(\nu\phi)}{[\cosh(\nu|\theta_+|) - \sin(\nu\phi)]^{1/2}[\cosh(\nu|\theta_-|) + \sin(\nu\phi)]^{1/2}}. \tag{115}$$

Thus, in general

$$|u_0(\phi)| \le \frac{\pi}{\Phi} \quad \text{for reactive faces}, \tag{116}$$

with equality for the rigid wedge ($\theta_+ = \theta_- = 0$).

The upper bound (116) may be replaced by an exponential bound in certain cases. For example, the identity (115) implies that the total field at the vertex of a narrow wedge-shaped region with reactive faces such that $\nu|\theta_\pm| \gg 1$ is exponentially small, and less than $4\nu \exp[-\frac{\nu}{2}(|\theta_+| + |\theta_-|)]$ in magnitude.

Pierce and Hadden [52] considered diffraction from a wedge with finite but large impedance, in which case approximations can be made using the rigid limit as reference. The same can be achieved for the tip field as follows. For hard faces the angles $\theta_\pm$ are small. For simplicity, we consider the case of identical impedances, $\theta_\pm = \theta$, and find after a simple expansion of (113) that

$$u_0(\phi) \approx \frac{\pi}{\Phi}\left[1 - \frac{\theta\nu}{\cos(\nu\phi)} + 2\theta\eta_\Phi\left(\phi - \Phi + \frac{\pi}{2}\right) - 2\theta\eta_\Phi\left(\phi + \Phi - \frac{\pi}{2}\right)\right]. \tag{117}$$

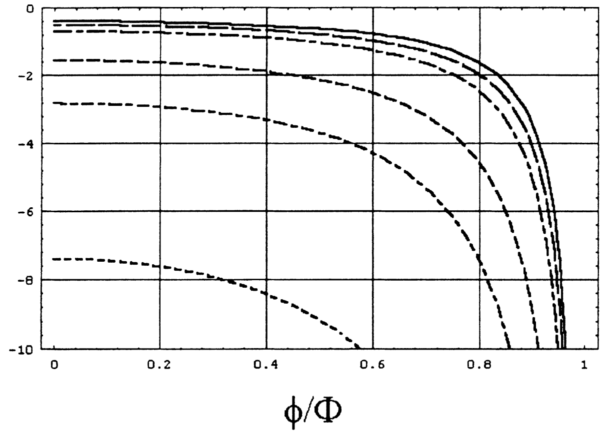

Fig. 5. Dependence of the edge field on the surface impedance for identical nearly hard boundaries.

The right-hand side may be simplified by using the integral representation (25)) for η_Φ and (28), to give

$$u_0(\phi) \approx \frac{\pi}{\Phi}\left\{1+\theta\left[\eta_{\Phi/2}\left(\phi+\frac{\pi}{2}\right)-\eta_{\Phi/2}\left(\phi-\frac{\pi}{2}\right)\right]\right\}. \tag{118}$$

The curves in Fig. 5 show the normalised coefficient of θ in Eq. (118), defined as $(\Phi/\pi)\partial u_0/\partial\theta|_{\theta=0}$, as a function of ϕ for different wedge angles: $\Phi = 7\pi/8$ (outermost), $3\pi/4$, $5\pi/8$, $3\pi/8$, $\pi/4$, and $\pi/8$ (innermost). We conclude that softening the faces, that is, increasing Re θ from 0, always has the effect of decreasing the tip field, and the rate of decrease is greatest for small wedge angles.

The function $u_0(\phi)$ in (118) considered as a function of $\phi \in (-\Phi, \Phi)$ has a single maximum at $\phi = 0$ and the stationary value is

$$u_0(0) = \frac{\pi}{\Phi}\left[1+2\theta\eta_{\Phi/2}\left(\frac{\pi}{2}\right)+\mathrm{O}(|\theta|^2)\right]. \tag{119}$$

The edge pressure depends upon the combination of Malyuzhinets functions given by the product $X_+(\phi,\theta_+)X_-(\phi,\theta_-)$, which as we saw above, is of unit magnitude if the resistive part of the impedances are zero. For general wedge angles the magnitudes $|X_\pm(\phi,\theta_\pm)|$ can be expressed, using Eqs. (30) and (34), as

$$\log|X_\pm(\phi,\theta_\pm)| = \sum_{l=1}^{\infty}\sum_{k=1}^{\infty}(-1)^{l+1}\log\left|\frac{[\phi\pm(\Phi-\pi/2-\theta_\pm)]^2-[\frac{\pi}{2}(2l-1)+2\Phi(2k-1)]^2}{[\phi\pm(\Phi-\pi/2+\theta_\pm)]^2-[\frac{\pi}{2}(2l-1)+2\Phi(2k-1)]^2}\right|. \tag{120}$$

Although this is a double sum, it easily programmed and converges quickly when the real parts of $\theta_\pm$ are small. Numerical tests have revealed the following chains of inequalities

$$|X_\pm(\phi,\theta)| \geq \left|X_\pm\left(\mp\Phi,\frac{\pi}{2}\right)\right| \geq \left|X_\pm\left(0,\frac{\pi}{2}\right)\right| = 0, \tag{121}$$

and

$$|X_\pm(\phi,\theta)| \leq \left|X_\pm\left(\pm\Phi,\frac{\pi}{2}\right)\right| \leq \left|X_\pm\left(\pm\pi,\frac{\pi}{2}\right)\right| \approx 2.07975. \tag{122}$$

These latter indicate that the magnitude of the ‘double-X’ function is of order unity, but never much more than one.

It is interesting to compare the magnitudes of the edge field relevant to a wedge and a flat impedance surface. To this end, consider a ratio

$$r = \left|\frac{u_0(0)}{u_0(0)|_{\Phi=\pi/2}}\right|, \tag{123}$$

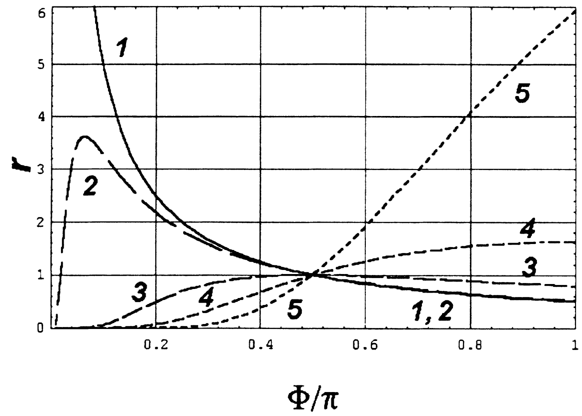

Fig. 6. Ratio of the edge field magnitudes relevant to a wedge and a flat surface of the same impedance as a function of Φ.

that relates the field $u_0(\phi_0)$ at the edge of a wedge with equal face impedances $\theta_+ = \theta_- = \theta$ to the one $u_0(\phi_0)|_{\Phi=\pi/2}$ on a flat surface with the same impedance, assuming that an incident plane wave comes from the direction $\phi_0 = 0$. Fig. 6 shows this ratio as a function of the vertex angle Φ for various values of the surface impedance: $\theta_\pm = 0$ (curve 1), $\theta_\pm = 0.05$ (curve 2), $\theta_\pm = \pi/2$ (curve 3), $\theta_\pm = 0.05 + 2.5\mathrm{i}$ (curve 4), $\theta_\pm = 0.05 + 5\mathrm{i}$ (curve 5).

The shapes of the calculated curves differ essentially according to the value of the impedance. For acoustically hard boundaries ($\theta = 0$), the parameter r grows without limit as $\Phi \to 0$, which clearly corresponds to the concentrating ability of a horn ($\Phi < \pi/2$) with ideal borders (curve 1). If a small absorption of the field energy by the horn walls is accounted for, then the dependence (123) becomes bounded and exhibits a maximum (curve 2), which means that for a given value of the impedance ($|\theta| \leq 1$) there exists an optimum value of the horn width at which a linear antenna placed at the edge would radiate most efficiently. Alternatively, in the case of substantially absorbing boundaries ($|\theta| > 1$) the shape of the computed curves changes to become a monotonically increasing function as the vertex angle Φ grows (curves 3 through 5) and the maximum is achieved at $\Phi = \pi$ (an impedance half-plane).

5. Conclusions

Malyuzhinets' theory for scattering from wedge boundaries combines many well known techniques from mathematical physics. These include: the Sommerfeld integral (2), which is the basic ansatz; the Laplace transform, which provides the inversion formula for the Sommerfeld integral and hence the crucial nullification theorem. The core of the Malyuzhinets theory involves functional difference equations, (52), which are solved using Fourier transforms and the fundamental Malyuzhinets function ψ_Φ. In this review we have attempted to emphasize the elegance and compactness of the Malyuzhinets theory, while remaining faithful to his original notation. In fact, his judicious choice of ψ_Φ as the central function has stood the test of time. The application of these results is now practical through the use of efficient computer algorithms for the ψ_Φ functions.

We have surveyed developments since the 1950s, when the bulk of Malyuzhinets' publications in this area appeared. In order to adequately describe the field behaviour near the edge of an impedance wedge, when $kr \leq 1$, the Malyuzhinets solution should be complemented by its alternative, series form. Thus, we have shown how to transform the Malyuzhinets integral into a sequence of Bessel functions. The original, integral form of the solution is best suited for the analysis of the field far from the edge of an impedance wedge, when $kr \gg 1$. Such analysis will be presented in our forthcoming publication.

The methods surveyed here are applicable not only to the problem considered, but have recently found novel application to diffraction from wedge boundaries with higher order boundary conditions. For detailed discussions of this subject see, for example, [80–86].

This paper has dealt with two dimensional fields applicable to the case when the incident wave falls at a right angle to the edge of the wedge, and consequently there is no dependence upon the third coordinate, say z, measured along the edge. A three dimensional extension of the Malyuzhinets solution for scattering of an obliquely incident plane wave from an impedance wedge is quite straightforward in acoustics. It can be achieved by separating out this dependence in the exponential factor $\exp(-\mathrm{i}kz\sin\chi_0)$ where χ_0 is the skewness angle and $\chi_0 = 0$ corresponds to normal incidence.

In contrast, the case of oblique or skew incidence in electromagnetics implies qualitative complications because of a need to work with vector Maxwell's equations rather than with a scalar Helmholtz equation. Generally, one has to solve a system of coupled equations for two unknown spectral functions and the solutions reported in the literature so far relate only to particular cases which are a wedge with surface impedance unity ($\theta_\pm = \pi/2$), a full plane impedance junction ($\Phi = \pi/2$), a half plane ($\Phi = \pi$), and right-angled exterior ($\Phi = 3\pi/4$) and interior ($\Phi = \pi/4$) wedges. The interested reader is referred to [73,87] for recent references and further discussion of this subject.

Similar complications arise for the elastic problem, in which the region $-\Phi < \phi < \Phi$ is described by the equations of isotropic dynamic elasticity, and the faces are either rigidly clamped or free of traction. The present analysis applies to the case of a shear wave polarised in the out-of-plane or z-direction. Any other polarization or any skew incidence of the shear wave leads to mode coupling and the problem once again becomes vectorial in nature, such that it can not be solved any more in an explicit form (see, for example, [66]).

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