

Exact complex source representations of time-harmonic radiation

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Abstract

New exact representations are presented for arbitrary compact source regions using distributions of sources in complex space. The equivalent sources are located on the complex “sphere” $r = ia$, with a weighting uniquely defined by the given far-field data. Each complex point source acts like a Gaussian beam with a well-defined directionality. This offers an exact wave-field representation suitable for arbitrary radiating fields, and optimal if the source is highly directional. In the limit of large ka the far-field pattern function is proportional to the associated weighting function for the complex sources. The theory is developed for both 2D and 3D, and numerical examples are presented for a 3D end-fire array.

1. Introduction

Gaussian beams are popular approximate solutions to the wave equation, as exemplified by the diverse use of the Gaussian beams summation technique in different fields, e.g., [1–4], and see [5,6] for reviews. The Gabor representation of data on a line or a plane [7,8] uses Gaussian functions as the basis set, and is an example of a windowed Fourier transform. However, the Gabor basis functions can only be propagated as approximate solutions of the wave equation. Complex point sources, which are closely related to Gaussian beams [9–12], have the useful property that they are exact wave fields. They have been used advantageously in modeling complex scattering phenomena that involve Gaussian beam incidence, e.g., [13,14].

Despite their obvious choice for modeling the very useful but simple case of a single Gaussian beam, complex point sources have not been widely used to represent other, directional wave fields. Recently, there has been some work in the direction [15–19]. For example, Boag and Mittra [15] develop new beam-like solutions to the vector-wave equation. However, these papers [15–19] do not provide a complete set of basis functions and one cannot expand a general field in terms of these beam-like solutions. The fundamental problem is the lack of an *exact* theory for representing arbitrary source distributions in terms of complex point sources. A suitable representation should be compact in space, like the source field it models, which rules out Gabor-type arrays of infinite extent [7,8]. In an earlier paper [20], one of the authors showed that the simple point source in free space can be replaced by a distribution of complex point sources on a sphere in complex space. In this paper we explore an extension of that

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approach, to find a way to replace a given but arbitrary radiating wave field by a distribution of complex point sources on a sphere.

Our purpose therefore is to derive new and exact representations for exterior fields in terms of complex point sources. We begin in Section 2 with a review of the standard representations using spherical and circular harmonic functions. The new complex source representation is given in Section 3 in terms of an unknown weighting function w which must be determined by solving an integral equation of the first kind. The exact solution of the integral equation is developed in Sections 4 and 5 for 3D and 2D problems, respectively. A numerical example is presented in Sections 6, where we assess the utility of a simple asymptotic solution to the 3D integral equation.

2. Introductory equations

2.1. Pattern function

We consider solutions to the Helmholtz equation regular outside of a source region in $\mathbb{R}^2$ or $\mathbb{R}^3$, bounded by the surface S with exterior domain V . Thus,

$$\nabla^2 f(\mathbf{x}) + k^2 f(\mathbf{x}) = 0 \quad \text{for } \mathbf{x} \text{ in } V, \quad (1)$$

and assuming the time dependence $e^{-i\omega t}$, the wave function $f(\mathbf{x})$ also satisfies the radiation condition

$$\frac{\partial f}{\partial r} - ikf = O(f/r), \quad r \rightarrow \infty, \quad (2)$$

where $r = |\mathbf{x}| \equiv (\mathbf{x} \cdot \mathbf{x})^{1/2}$. We note for later use that the branch of the square root is defined such that $R_e(\cdot)^{1/2} \geq 0$. The standard procedure is to represent the exterior field in terms of a set of solutions to the Helmholtz equation in separable coordinates. The spherical and circular representations are

$$f(\mathbf{x}) = \begin{cases} \sum_{n=0}^{\infty} h_n^{(1)}(kr) \sum_{m=-n}^n \bar{F}_{nm} Y_{nm}(\theta, \phi), & \text{in 3D,} \\ \sum_{n=-\infty}^{\infty} H_n^{(1)}(kr) \bar{F}_n e^{in\theta}, & \text{in 2D,} \end{cases} \quad (3a)$$

$$(3b)$$

where the spherical harmonics are

$$Y_{nm}(\theta, \phi) = \sqrt{\frac{(2n+1)}{4\pi} \frac{(n-m)!}{(n+m)!}} P_n^m(\cos \theta) e^{im\phi}. \quad (4)$$

The representation formulae in (3) are complete, and are, in principle, perfectly adequate for source fields measured on the surface of a sphere or circle. However, they are not particularly well suited to directional sources; with a main lobe in a specific direction. Many transducers are explicitly designed to have this property, which is at odds with the representation in terms of “global” angular functions, as in eq. (3). Transducer fields are sometimes well modeled by single or multiple Gaussian beams [21], which suggests that the optimal representation would be in terms of beam-like solutions, rather than spherical harmonic functions.

The wave function $f(\mathbf{x})$ in the exterior region is completely specified by the far-field pattern function F defined by

$$f(\mathbf{x}) = g(\mathbf{x}, 0) F(\hat{\mathbf{x}}), \quad kr \rightarrow \infty, \quad (5)$$

where $\hat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$, and $g(\mathbf{x}, y)$ is the free space Green's function,

$$\nabla^2 g + k^2 g = -\delta(\mathbf{x} - \mathbf{y}), \quad (6)$$

or explicitly,

$$g(\mathbf{x}, \mathbf{y}) = \begin{cases} \frac{ik}{4\pi} h_0^{(1)}(k|\mathbf{x} - \mathbf{y}|) = \frac{e^{ik|\mathbf{x} - \mathbf{y}|}}{4\pi|\mathbf{x} - \mathbf{y}|}, & \text{in 3D,} \\ \frac{i}{4} H_0^{(1)}(k|\mathbf{x} - \mathbf{y}|), & \text{in 2D.} \end{cases} \quad (7)$$

The pattern function follows from the standard representations in (3) as

$$F(\hat{\mathbf{x}}) = \begin{cases} \sum_{n=0}^{\infty} \sum_{m=-n}^n F_{nm} Y_{nm}(\theta, \phi), & \text{in 3D,} \\ \sum_{n=-\infty}^{\infty} F_n e^{in\theta}, & \text{in 2D,} \end{cases} \quad (8)$$

where

$$F_{nm} = k^{-1} 4\pi (-i)^{n+1} \bar{F}_{nm}, \quad F_n = 4(-i)^{n+1} \bar{F}_n. \quad (9a,b)$$

2.2. Complex point sources

A complex point source is the analytic extension of a Green's function into complex space. Let the source point be $\mathbf{y} = ia\mathbf{v}$ where a is real and positive, and $\mathbf{v}$ is a real unit vector. Define the related wave function which is weighted so that it behaves like a Gaussian beam in a particular direction,

$$G(\mathbf{x}, a\mathbf{v}) = e^{-ka} g(\mathbf{x}, ia\mathbf{v}). \quad (10)$$

The far-field form of G is then simply

$$G(\mathbf{x}, a\mathbf{v}) = g(\mathbf{x}, 0) e^{-ka(1 - \hat{\mathbf{x}} \cdot \mathbf{v})}, \quad kr \rightarrow \infty. \quad (11)$$

With $\hat{\mathbf{x}} \cdot \mathbf{v} = \cos \theta$, the pattern function of (10) is

$$F(\hat{\mathbf{x}}) = \exp \left\{ -2ka \sin^2 \frac{1}{2} \theta \right\}, \quad (12)$$

which has a definite lobe in the forward ($\theta = 0$) direction.

The connection between complex point sources and Gaussian beams follows from the near-axis approximation of the complex length $|\mathbf{x} - ia\mathbf{v}|$. Let $z = \mathbf{x} \cdot \mathbf{v}$ and $\rho = (r^2 - z^2)^{1/2}$, then for $|z - ia| \gg \rho$, or equivalently $z^2 + a^2 \gg \rho^2$,

$$|\mathbf{x} - ia\mathbf{v}| = -z - ia + \frac{\rho^2}{2(z - ia)} + \dots \quad (13)$$

The Gaussian beam approximation then follows from the further assumption that $ka \gg 1$, so that Eqs. (7), (10), and (13) imply to leading order that

$$G(\mathbf{x}, a\mathbf{v}) \approx \exp \left[ik \left(z + \frac{\rho^2}{2(z - ia)} \right) \right] \times \begin{cases} \frac{1}{4\pi(z - ia)}, & \text{in 3D,} \\ \frac{e^{i\pi/4}}{\sqrt{8\pi k(z - ia)}}, & \text{in 2D.} \end{cases} \quad (14)$$

Note that

$$\exp \left[ik \left(z + \frac{\rho^2}{2(z - ia)} \right) \right] = \exp \left[\frac{-ka\rho^2}{2(z^2 + a^2)} \right] \exp \left[ikz \left(1 + \frac{\rho^2}{2(z^2 + a^2)} \right) \right], \quad (15)$$

and hence the beam width is minimal at $z = 0$. The complex point source is thus closely related to a Gaussian beam, and for this reason we will refer to the fundamental “complex based Gaussian beam” of Eq. (10) as a CBGB.

3. Beam representation formula

Consider a solution of the Helmholtz equation which is regular outside a sphere of radius b . If $b \geq a$ then it is reasonable to expect that the wave function f can be represented by a distribution of complex point sources placed within the real sphere $r = a$. In particular, we imagine a distribution of CBGBs on the complex sphere $\mathbf{x} = ia\mathbf{v}$ where now $\mathbf{v}$ is any unit vector on the unit ball. Thus, we consider the ansatz

$$f(\mathbf{x}) = \int_{|\mathbf{v}|=1} G(\mathbf{x}, a\mathbf{v}) w(\mathbf{v}, ka) d\Omega(\mathbf{v}) \quad (16)$$

for some as yet undetermined function w . The far-field pattern for f may be obtained easily using Eqs. (11) and (16). The function f is regular outside $r = b \geq a$ and is therefore completely specified by its far-field pattern. Hence, by comparison with (5), it is clear that the weighting function w must satisfy the integral equation

$$\int_{|\mathbf{v}|=1} e^{-ka(1-\hat{\mathbf{x}} \cdot \mathbf{v})} w(\mathbf{v}, ka) d\Omega(\mathbf{v}) = F(\hat{\mathbf{x}}) \quad \text{for all } |\hat{\mathbf{x}}| = 1. \quad (17)$$

The problem, then, is to find the weighting function w which solves this identity. The 3D and 2D cases are considered separately below.

Norris [20] showed that a point source at the origin, i.e., $g(\mathbf{x}, 0)$, can be represented by an isotropic distribution of point sources on a complex sphere. This is equivalent to the general representation of Eq. (17) for the particular case of $F \equiv 1$. Referring to [20], we see that $w = \text{constant}$ in this case, with the explicit results (from [20, Eqs. (4) and (6)]),

$$f(\mathbf{x}) = g(\mathbf{x}, 0) \Leftrightarrow w = \begin{cases} \frac{e^{ka}}{4\pi j_0(ika)} = \frac{(ka)^{1/2} e^{ka}}{(2\pi)^{3/2} I_{1/2}(ka)}, & \text{in 3D,} \\ \frac{e^{ka}}{2\pi J_0(ika)} = \frac{e^{ka}}{2\pi I_0(ka)}, & \text{in 2D.} \end{cases} \quad (18)$$

Note that these weighting functions are purely real.

4. Applications in 3D

4.1. General solution of the integral equation

The direction $\hat{\mathbf{x}}$ is defined by the spherical polar angles θ and ϕ , and the pattern function $F(\theta, \phi)$ is given in Eq. (8). Let the angles θ' and ϕ' define the direction $\mathbf{v}$, and express the CBGB weighting function as

$$w(\theta', \phi') = \sum_{n=0}^{\infty} \sum_{m=-n}^n w_{nm} Y_{nm}(\theta', \phi'). \quad (19)$$

The integral equation (17) for w becomes

$$\sum_{n=0}^{\infty} \sum_{m=-n}^n F_{nm} Y_{nm}(\theta, \phi) = e^{-ka} \sum_{n=0}^{\infty} \sum_{m=-n}^n w_{nm} \int_{4\pi} e^{ka\hat{\mathbf{x}}(\theta', \phi') \cdot \mathbf{v}(\theta', \phi')} Y_{nm}(\theta', \phi') d\Omega(\theta', \phi'), \quad (20)$$

which must be satisfied for all $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$.

Using the identity [22, Eq. (11.3.46)]

$$e^{ikr\hat{\mathbf{x}}(\theta,\phi)\cdot\mathbf{v}(\theta',\phi')} = 4\pi \sum_{n=0}^{\infty} i^n j_n(kr) \sum_{m=-n}^n Y_{nm}(\theta, \phi) Y_{nm}^*(\theta', \phi'), \quad (21)$$

and the fact that spherical harmonics are, by definition, orthonormal with respect to integration over the angles, we get

$$\int_{4\pi} e^{ikr\hat{\mathbf{x}}(\theta,\phi)\cdot\mathbf{v}(\theta',\phi')} Y_{nm}(\theta', \phi') d\Omega(\theta', \phi') = 4\pi i^n j_n(kr) Y_{nm}(\theta, \phi). \quad (22)$$

Eqs. (20) and (22), together with the symmetry property $j_n(-z) = (-1)^n j_n(z)$, imply that

$$w_{nm} = \frac{i^n e^{ka}}{4\pi j_n(ika)} F_{nm} = \frac{(ka)^{1/2} e^{ka}}{(2\pi)^{3/2} I_{n+1/2}(ka)} F_{nm}. \quad (23)$$

It is worth noting that the ratios w_{nm}/F_{nm} are purely real.

The general expression for the weighting function follows from Eqs. (19) and (23) as

$$\begin{aligned} w(\theta, \phi) &= \frac{(ka)^{1/2} e^{ka}}{(2\pi)^{3/2}} \sum_{n=0}^{\infty} \frac{1}{I_{n+1/2}(ka)} \sum_{m=-n}^n F_{nn} Y_{nm}(\theta, \phi) \\ &= \frac{(ka)^{1/2} e^{ka}}{(2\pi)^{3/2}} \sum_{n=0}^{\infty} \sum_{m=-n}^n \frac{Y_{nm}(\theta, \phi)}{I_{n+1/2}(ka)} \int_{4\pi} F(\theta', \phi') Y_{nm}^*(\theta', \phi') d\Omega(\theta', \phi'). \end{aligned} \quad (24)$$

Inserting this result into the beam formula (16) and comparing with the spherical harmonics expansion (3) shows that the multipole functions $h_n^{(1)}(kr) Y_{nm}(\theta, \phi)$ can be expressed in terms of complex sources as

$$h_n^{(1)}(kr) Y_{nm}(\theta, \phi) = \frac{e^{ka}}{ik j_n(ika)} \int_{|\mathbf{v}|=1} Y_{nm}(\theta', \phi') G(\mathbf{x}, a\mathbf{v}) d\Omega(\mathbf{v}). \quad (25)$$

This generalizes the results of Norris [20] for a monopole source ($n = 0$) to arbitrary n . Note that it is quite distinct from the well-known concept of a multipole in complex space [15,23,24]. Here we are representing a multipole in *real* space by a distribution of monopoles in complex space.

4.2. Data on a scan sphere

Let $f_{nm}(r)$ be the expansion coefficients for the wave function on the sphere $r = \text{constant}$,

$$f(r, \theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^n f_{nm}(r) Y_{nm}(\theta, \phi). \quad (26)$$

Consider the reference value $r = b$, then

$$f(\mathbf{x}) = \sum_{n=0}^{\infty} \frac{h_n^{(1)}(kr)}{h_n^{(1)}(kb)} \sum_{m=-n}^n f_{nm}(b) Y_{nm}(\theta, \phi), \quad r \geq b, \quad (27)$$

and the far-field pattern follows from the asymptotic behavior of the spherical Hankel functions for large argument. The coefficients F_{nm} follow from Eqs. (3a), (9a) and (27), while (23) yields

$$w_{nm} = \left(\frac{2ka}{\pi} \right)^{1/2} \frac{(-i)^{n+1} e^{ka} f_{nm}(b)}{I_{n+1/2}(ka) h_n^{(1)}(kb)}, \quad (28)$$

and hence

$$w(\theta, \phi) = \left(\frac{2ka^{1/2}}{\pi} \right) \sum_{n=0}^{\infty} \frac{(-i)^{n+1} e^{ka}}{I_{n+1/2}(ka) h_n^{(1)}(kb)} \sum_{m=-n}^n f_{nm}(b) Y_{nm}(\theta, \phi), \quad a \leq b. \quad (29)$$

Using the asymptotic forms of the Bessel functions for large order it is found that the coefficients of (28) satisfy

$$|w_{nm}| \sim \text{constant} \cdot (b/a)^n n |f_{nm}(b)| \quad \text{as } n \rightarrow \infty. \quad (30)$$

Thus, the expansion for the weight function $w(\theta, \phi)$ converges slower than the expansion for the field on the sphere $r = b$, Eq. (26). Also, the fastest convergence of the expansion for w is obtained by choosing $a = b$. The same statement is true for the 2D case.

For example, if the surface data is a delta function,

$$f(b, \theta, \phi) = \frac{\delta(\theta - \theta_0)}{\sin \theta_0} \delta(\phi - \phi_0) = \sum_{n=0}^{\infty} \sum_{m=-n}^n Y_{nm}(\theta, \phi) Y_{nm}^*(\theta_0, \phi_0), \quad (31)$$

then

$$w(\theta, \phi) = \sum_{n=0}^{\infty} \frac{(-i)^{n+1} e^{ka} \sqrt{2ka/\pi}}{I_{n+1/2}(ka) h_n^{(1)}(kb)} \sum_{m=-n}^n Y_{nm}(\theta, \phi) Y_{nm}^*(\theta_0, \phi_0). \quad (32)$$

4.3. Approximate form of w for large ka

We consider the solution to the integral equation (17) for w in the asymptotic limit of $\lambda \gg 1$, where $\lambda = ka$. Let (θ_1, ϕ_1) be spherical polar coordinates about the observation direction $\hat{\mathbf{x}}$, so that $\hat{\mathbf{x}} \cdot \mathbf{v} = \cos \theta_1$, $d\Omega(\mathbf{v}) = \sin \theta_1 d\theta_1 d\phi_1$, and the integral equation becomes

$$F(\hat{\mathbf{x}}) = \int_0^{2\pi} d\phi_1 \int_0^{\pi} \sin \theta_1 d\theta_1 e^{-ka(1-\cos \theta_1)} w(\theta_1, \phi_1). \quad (33)$$

Let $\theta_1 = \lambda^{-1/2} \psi$, implying the asymptotic expansions

$$e^{-ka(1-\cos \theta_1)} \sin \theta_1 d\theta_1 = \frac{e^{-\psi^2/2}}{\lambda} d\psi \left[\psi + \frac{1}{\lambda} \left(\frac{\psi^5}{4!} - \frac{\psi^3}{3!} \right) + \frac{1}{\lambda^2} \left(\frac{\psi^9}{2(4!)^2} - \frac{\psi^7}{5!} + \frac{\psi^5}{5!} \right) + \dots \right]. \quad (34)$$

For simplicity we assume that the field and hence w and F are axially symmetric, and depend only on the spherical polar angle θ between $\hat{\mathbf{x}}$ and the axial direction $\hat{\mathbf{z}}$. That is, $F(\hat{\mathbf{x}}) = F(\theta)$. Therefore $w = w(\theta')$ in the integral equation (33), where θ' depends upon the given values of θ and the integration parameters θ_1 and ϕ_1 according to

$$\cos \theta' = \cos \theta \cos \theta_1 - \sin \theta \sin \theta_1 \cos \phi_1. \quad (35)$$

This may be solved in the neighborhood of θ by assuming an expansion for $\theta' - \theta$ in ascending powers of θ_1 and using $\theta_1 = \lambda^{-1/2} \psi$, resulting in the asymptotic expansion

$$\begin{aligned} \theta' = \theta &+ \frac{\psi}{\lambda^{1/2}} \cos \phi_1 + \frac{\psi^2}{2\lambda} \cot \theta \sin^2 \phi_1 - \frac{\psi^3}{3! \lambda^{3/2}} (1 + 3 \cot^2 \theta) \cos \phi_1 \sin^2 \phi_1 \\ &+ \frac{\psi^4}{4! \lambda^2} \cot \theta \sin^2 \phi_1 [\cos^2 \phi_1 (9 + 15 \cot^2 \theta) - 1 - 3 \cot^2 \theta] + \dots, \end{aligned} \quad (36)$$

which implies that

$$\begin{aligned}
\frac{1}{2\pi} \int_0^{2\pi} w(\theta') d\phi_1 &= w(\theta) + \frac{\psi^2}{4\lambda} (w''(\theta) + \cot \theta w'(\theta)) \\
&+ \frac{\psi^4}{64\lambda^2} [w''''(\theta) + 2 \cot \theta w'''(\theta) - (4 + \cot^2 \theta) w''(\theta) \\
&+ \cot \theta (\frac{5}{3} + \cot^2 \theta) w'(\theta)] + \dots
\end{aligned} \tag{37}$$

Combining (33), (34), and (37), the integral equation (17) becomes

$$\begin{aligned}
F(\theta) &= \frac{2\pi}{\lambda} \int_0^{\pi\lambda^{1/2}} e^{-\psi^2/2} d\psi \left\{ \psi w(\theta) + \frac{1}{\lambda} \left[\frac{\psi^3}{4} (w''(\theta) + w'(\theta) \cot \theta) + \left(\frac{\psi^5}{4!} - \frac{\psi^3}{3!} \right) w(\theta) \right] \right. \\
&+ \frac{1}{\lambda^2} \left[\frac{\psi^5}{64} [w''''(\theta) + 2 \cot \theta w'''(\theta)] + [\psi^7 - \psi^5 (6 + \frac{3}{2} \cot^2 \theta)] \frac{w''(\theta)}{96} \right. \\
&\left. \left. + \left[\frac{\psi^7}{96} + \frac{\psi^5}{64} (\cot^2 \theta - 1) \right] \cot \theta w'(\theta) + \left(\frac{\psi^9}{2(4!)^2} - \frac{\psi^7}{5!} + \frac{\psi^5}{5!} \right) w(\theta) \right] + \dots \right\}.
\end{aligned} \tag{38}$$

The upper limit of integration can be replaced by ∞ with no change in the asymptotic expansion, but simplifying the integrals, to give

$$\begin{aligned}
\frac{\lambda}{2\pi} F(\theta) &= w(\theta) + \frac{1}{2\lambda} (w''(\theta) + \cot \theta w'(\theta)) \\
&+ \frac{1}{8\lambda^2} [w''''(\theta) + 2 \cot \theta w'''(\theta) - \cot^2 \theta w''(\theta) + (3 + \cot^2 \theta) \cot \theta w'(\theta)] + O\left(\frac{1}{\lambda^2}\right).
\end{aligned} \tag{39}$$

We assume the asymptotic expansion

$$w = \frac{\lambda}{2\pi} \left(w_0 + \frac{1}{\lambda} w_1 + \frac{1}{\lambda^2} w_2 + O(\lambda^{-3}) \right), \tag{40}$$

then substitute this into (39) and equate the terms of order λ , λ^0 and λ^{-1} , etc. This gives a sequence of equations:

$$w_0 = F, \tag{41}$$

$$w_1 = -\frac{1}{2} (w_0'' + \cot \theta w_0'), \tag{42}$$

$$w_2 = -\frac{1}{8} [w_0'''' + 2 \cot \theta w_0''' - \cot^2 \theta w_0'' + (3 - \cot^2 \theta) \cot \theta w_0'] - \frac{1}{2} (w_1'' + \cot \theta w_1'), \tag{43}$$

which may be solved recursively. Finally, we obtain the asymptotic expansion of the solution to the integral equation as

$$\begin{aligned}
w(\theta, ka) &= \frac{ka}{2\pi} F(\theta) - \frac{1}{4\pi} (F''(\theta) + \cot \theta F'(\theta)) \\
&+ \frac{1}{16\pi ka} [F''''(\theta) + 2 \cot \theta F'''(\theta) - (3 + 2 \cot^2 \theta) F''(\theta) + (3 \cot^2 \theta - 1) \cot \theta F'(\theta)] \\
&+ O\left(\frac{1}{(ka)^2}\right).
\end{aligned} \tag{44}$$

The far-field pattern function F also depends upon k , but not a . Therefore, the asymptotic expansion (44) may or may not be valid for large frequencies, depending how the derivatives of F behave in that limit. However, Eq. (44) is asymptotically correct for large a and fixed k .

5. Applications in 2D

5.1. General solution for w

Let θ be the polar angle defining $\hat{\mathbf{x}}$ such that the far-field pattern $F(\theta)$ can be represented as the Fourier series in (8). The integral equation (17) is now

$$\sum_{n=-\infty}^{\infty} F_n e^{in\theta} = \sum_{m=-\infty}^{\infty} w_m \int_0^{2\pi} e^{-ka(1-\cos(\psi-\theta))} e^{im\psi} d\psi, \quad (45)$$

where the unknown distribution function is

$$w(\theta) = \sum_{n=-\infty}^{\infty} w_n e^{in\theta}. \quad (46)$$

A simple change of variables and the orthogonality of the functions $e^{in\theta}$, combined with the identity,

$$\int_0^{2\pi} e^{in\phi + ka \cos \phi} d\phi = 2\pi (-i)^n J_n(ika) = 2\pi I_n(ka), \quad (47)$$

implies that

$$w_n = \frac{e^{ka} F_n}{2\pi I_n(ka)}. \quad (48)$$

Note that the ratio w_n/F_n is purely real.

The general expression for the weighting function is therefore

$$w(\theta) = \frac{e^{ka}}{2\pi} \sum_{n=-\infty}^{\infty} \frac{F_n e^{in\theta}}{I_n(ka)} = \frac{e^{ka}}{4\pi^2} \sum_{n=-\infty}^{\infty} \frac{e^{in\theta}}{I_n(ka)} \int_0^{2\pi} F(\theta') e^{-in\theta'} d\theta', \quad (49)$$

and the two-dimensional CBGB representation for the radiated field is

$$f(\mathbf{x}) = \frac{e^{ka}}{2\pi} \sum_{n=-\infty}^{\infty} \frac{F_n}{I_n(ka)} \int_0^{2\pi} G(\mathbf{x}, a\mathbf{v}(\psi)) e^{in\psi} d\psi. \quad (50)$$

A complex source representation for the 2D multipole $H_n^{(1)}(kr) e^{in\theta}$ follows from Eqs. (3), (9) and (50) as

$$H_n^{(1)}(kr) e^{in\theta} = \frac{4e^{ka}}{iJ_n(ika)} \int_0^{2\pi} G(\mathbf{x}, a\mathbf{v}(\psi)) e^{in\psi} d\psi. \quad (51)$$

Again, this is the generalized version of Norris' [20] 2D monopole representation.

5.2. Example 1: Data on a circle

Let $f = f(r, \theta)$, and let $f_n(r)$ be its Fourier coefficients:

$$f(r, \theta) = \sum_{n=-\infty}^{\infty} f_n(r) e^{in\theta} \Leftrightarrow f_n(r) = \bar{F}_n H_n^{(1)}(kr) \quad (52)$$

for all $r > r_{\text{source}}$ where r_{source} is the radius of the smallest circle enclosing all the sources. Suppose that the radiating wave function $f(\mathbf{x})$ is defined by its value on $r = b$, then

$$f(\mathbf{x}) = \sum_{n=-\infty}^{\infty} f_n(b) \frac{H_n^{(1)}(kr)}{H_n^{(1)}(kb)} e^{in\theta}, \quad r \geq b, \quad (53)$$

and the far-field pattern follows from the asymptotic behavior of the Hankel functions for large argument, i.e.,

$$F_n = \frac{4(-i)^{n+1}}{H_n^{(1)}(kb)} f_n(b), \quad (54)$$

and consequently

$$w(\theta) = \frac{2e^{ka}}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-i)^{n+1} e^{in\theta} f_n(b)}{I_n(ka) H_n^{(1)}(kb)}. \quad (55)$$

For instance, consider a delta function on the circle, $f(b, \theta) = \delta(\theta - \theta_0)$, for which the CBGB weighting function is

$$w(\theta) = \frac{e^{ka}}{\pi^2} \sum_{n=-\infty}^{\infty} \frac{(-i)^{n+1} e^{in(\theta - \theta_0)}}{I_n(ka) H_n^{(1)}(kb)}. \quad (56)$$

5.3. Example 2: Gaussian data on a line

Consider initial data on the line $\mathbf{x} \cdot \mathbf{v} = 0$ like a Gaussian function centered at the origin:

$$f(\mathbf{x}) = A e^{-kr^2/2a}, \quad \frac{\partial f(\mathbf{x})}{\partial n} = ikA e^{-kr^2/2a}, \quad \text{on } \mathbf{x} \cdot \mathbf{v} = 0, \quad (57)$$

where $A = i/\sqrt{8\pi ka}$ and $\mathbf{v}$ is a fixed real unit vector. The source in this case is not confined to a finite source region. Furthermore, it is not possible to arbitrarily prescribe both f and $\partial f/\partial n$, as in (57), and to simultaneously satisfy the wave equation. The correct procedure calls for one of these to be given, and the other follows from the Helmholtz integral equation

$$f(\mathbf{x}) = \int_S \left\{ f(\mathbf{y}) \frac{\partial g(\mathbf{x}, \mathbf{y})}{\partial n(\mathbf{y})} - g(\mathbf{x}, \mathbf{y}) \frac{\partial f(\mathbf{y})}{\partial n(\mathbf{y})} \right\} dS(\mathbf{y}) \quad \text{for } \mathbf{x} \text{ in } V, \quad (58)$$

where $\mathbf{n}$ is directed into V . The far-field of f follows from Eq. (58) and the far-field expansion of Green's function:

$$g(\mathbf{x}, \mathbf{y}) = g(\mathbf{x}, 0) e^{ik\hat{\mathbf{x}} \cdot \mathbf{y}}, \quad kr \rightarrow \infty. \quad (59)$$

Thus, from Eqs. (5), (58), and (59),

$$F(\hat{\mathbf{x}}) = - \int_S \left\{ ik\hat{\mathbf{x}} \cdot \mathbf{n} f(\mathbf{y}) + \frac{\partial f(\mathbf{y})}{\partial n(\mathbf{y})} \right\} e^{ik\hat{\mathbf{x}} \cdot \mathbf{y}} dS(\mathbf{y}). \quad (60)$$

Substituting (57) into Eq. (60) yields

$$F(\hat{\mathbf{x}}) = \cos^2 \frac{1}{2} \theta e^{-(1/2)ka \sin^2 \theta}, \quad (61)$$

where $\cos \theta = \hat{\mathbf{x}} \cdot \mathbf{v}$. The pattern function (61) is very similar to that of a complex point source, see Eq. (12).

As an example, consider the Gaussian of equation (57) with far-field pattern given by Eq. (61). Using the identity (47), it can be shown that

$$F_n = \begin{cases} \frac{e^{-ka/4}}{2} I_{n/2}(ka/4), & n \text{ even}, \\ \frac{e^{-ka/4}}{4} [I_{(n-1)/2}(ka/4) + I_{(n+1)/2}(ka/4)], & n \text{ odd}. \end{cases} \quad (62)$$

and the weighting function follows from Eqs. (46) and (48). Using Eq. (49), we obtain

$$w(\theta) = \frac{e^{3ka/4}}{4\pi} \sum_{n \text{ even}} \frac{I_{n/2}(ka/4)}{I_n(ka)} e^{in\theta} + \frac{e^{3ka/4}}{8\pi} \sum_{n \text{ odd}} \left[\frac{I_{(n-1)/2}(ka/4)}{I_n(ka)} + \frac{I_{(n+1)/2}(ka/4)}{I_n(ka)} \right] e^{in\theta}. \quad (63)$$

This is the weighting function required to produce an *exact* radiated field corresponding to the “data” of Eq. (57). We would expect that it closely approximates a delta function in the limit of large ka . Letting $ka \rightarrow \infty$ and using the asymptotic formulae for the modified Bessel functions, we find that $w_n \rightarrow 1/2\pi$ and hence $w(\theta) \rightarrow \delta(\theta)$.

5.4. Approximate form of w for large ka

Eq. (17) becomes in 2D,

$$F(\theta) = \int_{-\pi}^{\pi} e^{-ka(1-\cos \theta')} w(\theta - \theta', ka) d\theta'. \quad (64)$$

We again consider this integral equation for w in the asymptotic limit of $\lambda \gg 1$, where $\lambda = ka$. Let $\theta' = \lambda^{-1/2}\psi$, so that we have the asymptotic expansions

$$e^{-ka(1-\cos \theta')} = e^{-\psi^2/2} \left(1 + \frac{\psi^4}{\lambda 4!} - \frac{\psi^6}{\lambda^2 6!} + \frac{\psi^8}{\lambda^2 2(4!)^2} + O(\lambda^{-3}) \right), \quad (65)$$

$$w(\theta - \theta') = w(\theta) - \frac{\psi}{\lambda^{1/2}} w'(\theta) + \frac{\psi^2}{2\lambda} w''(\theta) - \frac{\psi^3}{\lambda^{3/2} 3!} w'''(\theta) + \frac{\psi^4}{\lambda^2 4!} w''''(\theta) + O(\lambda^{-5/2}). \quad (66)$$

Substituting these into (64) yields

$$F(\theta) = \frac{1}{\lambda^{1/2}} \int_{-\pi \lambda^{1/2}}^{\pi \lambda^{1/2}} d\psi e^{-\psi^2/2} \left[w(\theta) + \frac{1}{\lambda} \left(\frac{\psi^2}{2} w''(\theta) + \frac{\psi^4}{4!} w(\theta) \right) + \frac{1}{\lambda^2} \left(\frac{\psi^4}{4!} w''''(\theta) + \frac{\psi^6}{2(4!)} w''(\theta) + \left(\frac{5}{8} \psi^8 - \psi^6 \right) \frac{w(\theta)}{6!} \right) + O(\lambda^{-3}) \right], \quad (67)$$

where we have simplified by using the fact that the integrals of odd powers of ψ vanish. The remaining integrals can be expressed in terms of error functions, and then expanded asymptotically in λ . The net result is that we may replace the limits of integration in (67) by $\pm\infty$ because the additional asymptotic terms are exponentially small, and so obtain

$$\begin{aligned} \frac{\lambda^{1/2}}{(2\pi)^{1/2}} F(\theta) = w(\theta) + \frac{1}{\lambda} \left(\frac{1}{2} w''(\theta) + \frac{1}{8} w(\theta) \right) \\ + \frac{1}{\lambda^2} \left(\frac{1}{8} w''''(\theta) + \frac{5}{16} w''(\theta) + \frac{9}{128} w(\theta) \right) + O(\lambda^{-3}). \end{aligned} \quad (68)$$

Now assume the asymptotic expansion

$$w = \frac{\lambda^{1/2}}{(2\pi)^{1/2}} \left(w_0 + \frac{1}{\lambda} w_1 + \frac{1}{\lambda^2} w_2 + O(\lambda^{-3}) \right). \quad (69)$$

By substituting this into (68) and equating the terms of order $\lambda^{1/2}$, $\lambda^{-1/2}$ and $\lambda^{-3/2}$ sequentially, yields

$$w_0 = F, \quad (70)$$

$$w_1 = -\frac{1}{2} w_0'' - \frac{1}{8} w_0, \quad (71)$$

$$w_2 = -\frac{1}{2} w_1'' - \frac{1}{8} w_1 - \frac{1}{8} w_0'''' - \frac{5}{16} w_0'' - \frac{9}{128} w_0, \quad (72)$$

which may be solved iteratively. Finally, we obtain the asymptotic expansion

$$\begin{aligned} w(\theta, ka) = \sqrt{\frac{ka}{2\pi}} \left\{ F(\theta) - \frac{1}{2ka} \left(F''(\theta) + \frac{1}{4} F(\theta) \right) \right. \\ \left. + \frac{1}{8(ka)^2} \left(F''''(\theta) - \frac{3}{2} F''(\theta) - \frac{7}{16} F(\theta) \right) \right\} + O\left(\frac{1}{(ka)^{5/2}}\right). \end{aligned} \quad (73)$$

The weighting function w for large ka is again simply proportional to the far-field pattern.

6. Numerical example in 3D

The 3D beam-representation (16) will now be verified numerically by computing the far-field pattern of a collection of 11 point sources. Moreover, we shall investigate the accuracy of the asymptotic approximation to the weighting function.

The point sources are located on the z -axis at $\mathbf{x}_q = (-\lambda + q \Delta z) \hat{\mathbf{z}}$ where $q = 0, 1, 2, \dots, 10$, λ is the wavelength, and $\Delta z = \frac{1}{5} \lambda$. The phase change of $-k \Delta z$ between adjacent point sources just offsets the propagation-phase advance from point source to point source when the radiation is computed in the negative z direction. An array of this type is called an end-fire array and has one main beam directed in the negative z direction. The field from this collection of point sources is given by

$$f(\mathbf{x}) = \sum_{q=0}^{10} g(\mathbf{x}, \mathbf{x}_q) e^{iqk \Delta z}, \quad (74)$$

where $g(\mathbf{x}, \mathbf{x}_q)$ is the free-space Green's function (7). The corresponding far-field pattern is thus a function of θ only:

$$F(\theta, \cdot) = \sum_{q=0}^{10} e^{-ik(-\lambda + q \Delta z) \cos \theta} e^{-iqk \Delta z} = \left(\frac{1 - e^{-i11k \Delta z(1 + \cos \theta)}}{1 - e^{-ik \Delta z(1 + \cos \theta)}} \right) e^{ik \lambda \cos \theta}. \quad (75)$$

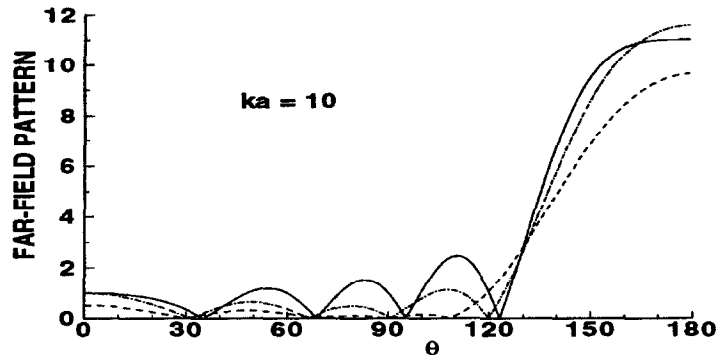


Fig. 1. The magnitude $|F(\theta)|$ of the far-field pattern of the array containing 11 point sources with $ka = 10$. The exact result of (75) and the beam formula of (77) are indistinguishable and are shown by the solid curve (—). The dashed curve (---) shows the result of the beam formula (77) with the single term asymptotic approximation to w from (44). The result of using a two-term asymptotic expansion is given by the dash-dot curve (— · —).

To compute the ϕ -independent weighting function $w(\theta, \cdot)$, we shall use the formula (29) involving the field on the scan sphere $r = b$. We have $f_{nm}(b) = 0$ for $m \neq 0$ and

$$f_{n0}(b) = 2\pi \int_0^\pi f(b(\sin \theta \hat{x} + \cos \theta \hat{z})) Y_{n0}^*(\theta, \cdot) \sin \theta d\theta. \quad (76)$$

The number of terms needed to accurately compute the weighting function in the summation (29) is determined by the formula $N = [kr_s] + n_1$ where r_s is the radius of the source region and n_1 is a small integer [25, p.17]. In the present case $r_s = \lambda$ and thus $N = 12$ is sufficiently large. One can then use the formulae of Hansen [26, Section 4.1] to compute $f_{n0}(b)$ for $n = 0, 1, 2, \dots, 12$ from the values of the field (74) on the scan sphere $r = b$ at the angles $\theta = n\Delta\theta$ with $n = 0, 1, 2, \dots, 12$, and $\Delta\theta = 2\pi/25$.

In the following numerical calculations we choose the radius of the scan sphere equal to the radius of the sphere that encloses the complex point sources in the beam expansion, that is $b = a$. In this case the far-field pattern is ϕ independent and it is found from (17) that

$$F(\theta, \cdot) = 2\pi \int_0^\pi e^{-ka(1 - \cos \theta \cos \theta')} w(\theta', \cdot) I_0(ka \sin \theta \sin \theta') \sin \theta' d\theta', \quad (77)$$

where we have used the integral representation (47) for the modified cylindrical Bessel function.

Now to the numerical results. Figs. 1 and 2 compare the exact far-field pattern with three alternative descriptions: (i) the beam formula (77) based on the complex source weighting function determined from (76); (ii) the beam formula using the first term of the asymptotic expansion (44) for w ; and (iii) the same but using the first two terms in the asymptotic expansion. The result (i) obtained from the beam formula (77) is seen to coincide with the exact far-field pattern; whereas the simple asymptotic result (ii) does not agree well with the exact far-field pattern for $ka = 10$. The agreement improves for $ka = 40$, but in either case, the two-term approximation (iii) to w is far better.

In order to see which Gaussian beams contribute to the far field, Figs. 3 and 4 show the integrand of (77) for three values of θ . Fig. 3 for $ka = 10$ shows that the integrand is not well localized around the particular values of θ , and hence the simple, one-term approximation to w cannot be expected to work well. In contrast, at $ka = 40$ Fig. 4 indicates that the integrand is now quite well localized around the particular value of θ . It is interesting to note that

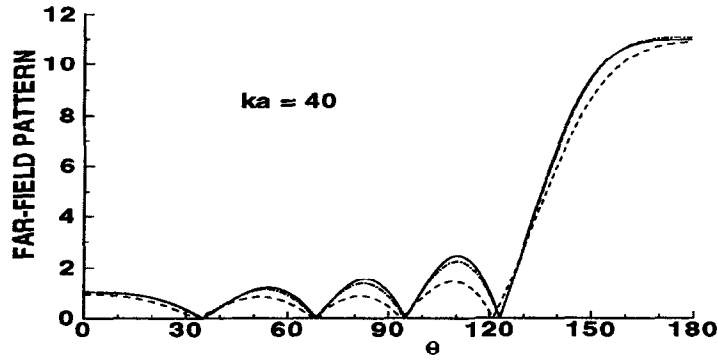


Fig. 2. The same as Fig. 1 but for $ka = 40$.

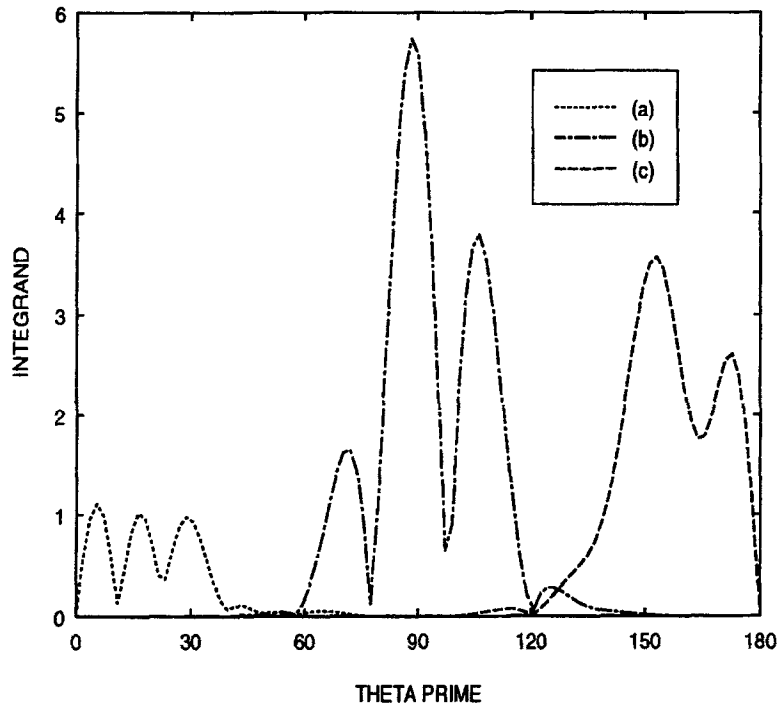


Fig. 3. The integrand of the ϕ -independent beam integral with $ka = 10$ for: (a) $\theta = 0^\circ$; (b) $\theta = 90^\circ$; (c) $\theta = 180^\circ$.

even though the integrand is not well localized for $ka = 10$, see Fig. 3, the two-term asymptotic expansion captures some of the non-local behavior.

7. Conclusion

Complex point sources serve as useful building bricks for modeling radiating wave fields. Although they are closely related to Gaussian beams, and can reproduce the radiation pattern of a Gaussian transducer very well, the

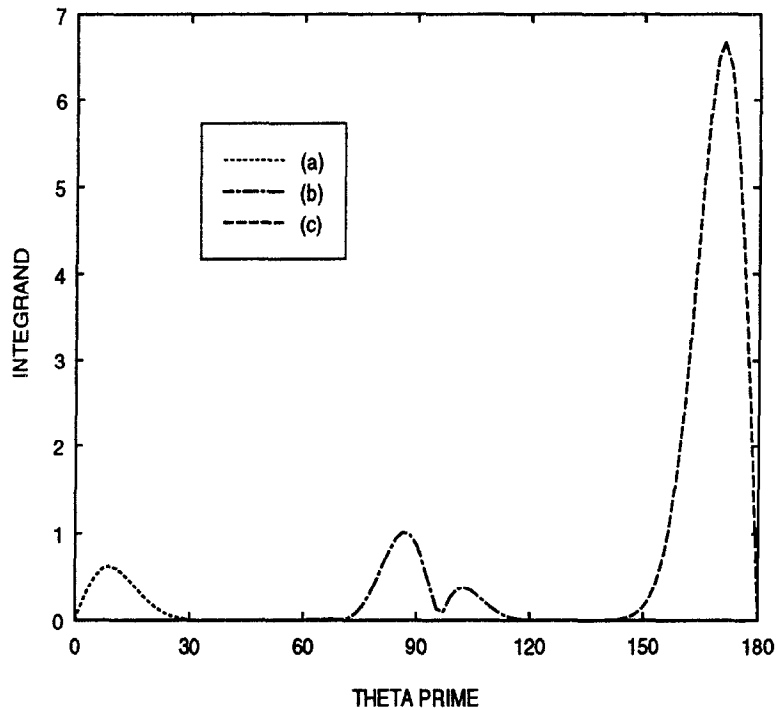


Fig. 4. The same as Fig. 3 but for $ka = 40$.

present results show that they are not restricted to these types of radiation patterns. We have shown that an arbitrary radiation pattern from a compact region can be replicated *exactly* by a compact distribution of complex point sources. The main results are the formulae for the weighting function w in terms of the radiated field on a sphere or circle in real space, for example, Eqs. (29) and (55) for 3D and 2D, respectively. The weighting function also depends upon the arbitrary value for the distance a representing the complex source radius, which could possibly be used to advantage. Thus, the asymptotic expansions for w in Eqs. (44) and (73) offer explicit, local expressions for the source density in terms of the far-field pattern function. In closing, we believe the present results, exact and asymptotic alike, offer a completely new approach to wave-field modeling. Applications to scattering, as well as extensions of the theory to the time domain [27,28] will be examined elsewhere.

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