

Ray tracing over smooth elastic shells of arbitrary shape

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An efficient numerical scheme based on ray theory is developed for the analysis of elastic waves traveling over a fluid-loaded smooth elastic shell of arbitrary shape. The shell's surface is first discretized into a number of small patches. The local geometry of each patch is then approximated in a parametric form using bi-cubic spline functions. A local curvilinear coordinate frame is defined on each patch. The ray trajectories and ray-tube areas are obtained by solving a set of ordinary differential equations, the ray and transport equations, within each patch. Several numerical tests of the accuracy and efficiency of the scheme were carried out on spherical and ellipsoidal elastic shells. The numerical results for the spherical shell agree well with analytical solutions. The ray trajectories and the ray-tube areas over an ellipsoidal shell with three distinct semiaxes clearly illustrate the inhomogeneous and anisotropic effects due to the variable curvature on the shell's surface. It is also observed numerically that the magnitude of the ray-tube area along a ray is directly correlated with the stability of its trajectory. © 1996 Acoustical Society of America.

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INTRODUCTION

Many fundamental structural acoustic problems are most naturally and easily interpreted in terms of rays. This is particularly appropriate at high frequencies where the energy flows predominantly along rays. The ray point of view has lead to enormous insight into the mechanism of wave propagation on fluid-loaded elastic shells.¹⁻³ Our main interest here is to understand the mechanism of elastic wave propagation on fluid-loaded smooth shells of arbitrary shape using ray theory. Complete understanding of this issue is essential to the description of acoustic radiation and scattering from shells surrounded by fluid.

Ray concepts taken from geometrical optics⁴ have been successfully applied to the investigations of wave propagation on spherical and cylindrical acoustic or elastic shells.^{1,5-12} Attempts to extend the application of ray concepts to arbitrarily curved surfaces can be traced back to the works of Ross,¹³ Steele,¹⁴ and Germogenova.¹⁵ Ross developed high-frequency solutions in the form of asymptotic series for shell dynamics in the absence of fluid loading. Using similar methods Steele¹⁴ reduced the partial differential equations of thin elastic shells to a system of ordinary differential equations, i.e., ray and transport equations. Most recently, the geometric ideas of these authors were further exploited and elaborated by Pierce,¹⁶ and Norris and Rebinsky,^{17,18} with explicit consideration of fluid loading. Pierce derived a general dispersion relation in his paper¹⁶ for fluid-loaded shells of arbitrary shape. Norris and Rebinsky

decomposed Pierce's general dispersion relation into that associated only with each distinct wave type and expressed the ray and transport equations in relatively simple forms.^{18,19} In addition, they derived general results for the interaction of an acoustic field with a smooth thin shell submerged in a fluid.¹⁷ The ray-type representations developed by these authors are valid for arbitrary smooth shells of variable curvature. However, most of the applications of ray methods to date are limited to shells of simple regular geometry such as spheres, cylinders, and cones. The main difficulty encountered in practical applications is the lack of a global explicit expression for the geometry of arbitrarily curved shells, and consequently the ray and transport equations cannot be solved numerically in a systematic manner.

Our objective here is to bridge the gap between the ray theories of shell dynamics and their practical applications to shells of arbitrary shape, and to develop an efficient ray-based numerical scheme. The principal assumption used is that the elastic wavelengths are much greater than the shell thickness but much smaller than the principal radii of curvature of the shell. The paper is organized as follows. The ray theories derived by previous authors, in particular, by Norris and Rebinsky,^{17,18} are briefly summarized in Sec. I. A local parametric representation for the shell surface geometry is introduced in Sec. II. A numerical scheme is then developed to trace rays over an arbitrarily curved, smooth surface by solving ray and transport equations. A number of numerical examples are presented in Sec. III. The efficiency and accuracy of the scheme are examined and comparisons are made with the available analytical solutions.

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I. THEORETICAL PREMISES

We first summarize some well known theoretical results and define the notations used in our numerical scheme.

A. Description of the shell geometry

The geometry of an arbitrarily curved shell is best described in terms of its differential or local geometry.²⁰ The position vector of the point at (ξ^1, ξ^2) on the surface Σ is written as

$$\mathbf{x} = \mathbf{x}(\xi^1, \xi^2), \quad (1)$$

where ξ^1 and ξ^2 are curvilinear coordinates. We adopt the conventional tensor notation used in mechanics, by, for example, Green and Zerna²¹ and Pierce.²² Sub- and superscript indices denote the covariant and contravariant tensor forms, respectively. The greek indices assume the values 1 or 2, and the suffix, α or the symbol ∇_α means differentiation with respect to ξ^α . The direction vectors associated with ξ^1 curves and ξ^2 curves are defined as

$$a_\alpha = \nabla_\alpha \mathbf{x} = \mathbf{x}_{,\alpha}, \quad \alpha = 1, 2. \quad (2)$$

The surface metric and curvature tensors can be uniquely defined for this curvilinear coordinate system, we refer to the Appendix for details.

We note that it is often difficult to establish one global curvilinear coordinate system on a surface of arbitrary shape. Our approach will be to define a number of different local curvilinear coordinate frames throughout the shell's surface, such that Eqs. (1) and (2) are defined continuously within each distinct "patch."

B. Thin shell ray equations

Our starting point is the leading order asymptotic approximation to the membrane wave dispersion relation on a thin shell under fluid loading. A membrane wave is a shell wave with motion mainly in the tangential plane of the shell. Consider such a wave propagating in the direction specified by a unit vector $\mathbf{u}$ at a point on the shell, with harmonic time dependence $e^{-i\omega t}$. The local wave number k satisfies^{18,19}

$$k^2 \approx \frac{\omega^2}{c^2} - \frac{Z_m}{Z_m + Z_s} \frac{1}{R_0^2}, \quad (3)$$

where c is the phase speed associated with waves on a flat plate: $c = c_p$ for longitudinal waves, or $c = c_t$ for transverse waves. The longitudinal and transverse plate wave speeds are $c_p = \sqrt{C/m}$ and $c_t = c_p \sqrt{(1-\nu)/2}$, where $C \equiv Eh/(1-\nu^2)$ is the extensional stiffness, E is the Young's modulus, ν is the Poisson's ratio, h is the thickness, and m is the mass density per unit area of the shell. The impedances in Eq. (3) are

$$Z_m = -i\omega m, \quad Z_s = \rho_f c_f \sec \theta_0, \quad (4)$$

where $\rho_f c_f$ is the fluid acoustic impedance and $\theta_0 = \sin^{-1}(c_f/c)$. The remaining parameter in Eq. (3) is the dynamic effective local radius of curvature, R_0 , defined by

$$\frac{1}{R_0} = \begin{cases} \frac{1}{R_\parallel} + \frac{\nu}{R_\perp}, & \text{longitudinal,} \\ \frac{2}{R_T}, & \text{transverse,} \end{cases} \quad (5)$$

with

$$\begin{aligned} \frac{1}{R_\parallel} &= u^\alpha d_{\alpha\beta} u^\beta, & \frac{1}{R_\perp} &= u^\perp{}^\alpha d_{\alpha\beta} u^\perp{}^\beta, \\ \frac{1}{R_T} &= \frac{1}{2} (u^\alpha d_{\alpha\beta} u^\perp{}^\beta + u^\perp{}^\alpha d_{\alpha\beta} u^\beta), \end{aligned} \quad (6)$$

where $d_{\alpha\beta}$ are the covariant components of the curvature tensor defined by Eq. (A2), and $\mathbf{u}^\perp$ is a unit vector on Σ perpendicular to the ray direction $\mathbf{u}$. Note that the final term in (3) is complex. The imaginary part of this term accounts for the attenuation of membrane waves through radiation loss to the ambient fluid.

The characteristic curves of the eikonal Eq. (3) define the ray trajectories.¹⁹ However, to a first approximation we can ignore the influence of the fluid-loading term, which is small, and instead consider the simpler eikonal equation

$$k^2 = \omega^2/c^2, \quad (7)$$

where k is the magnitude of the surface wave-number vector, $\mathbf{k}$, with components

$$k_\alpha = \nabla_\alpha \phi = k u_\alpha.$$

Here, $\phi(\mathbf{x})$ is the phase function on the surface Σ . The characteristic curves of the eikonal equation (7) are families of curves on Σ parametrized by their arc length s . The components of the position vectors along these curves in the curvilinear coordinate system, denoted by (ξ^1, ξ^2) , must satisfy the differential equations¹⁹

$$\frac{d\xi^\alpha}{ds} = u^\alpha \quad (8)$$

and

$$\frac{du^\alpha}{ds} + \Gamma_{\beta\gamma}^\alpha u^\beta u^\gamma + u^\perp{}^\alpha u^\perp{}^\beta \frac{c_{,\beta}}{c} = 0, \quad (9)$$

where, u^α , $\alpha=1,2$, are the contravariant components of the unit wave normal of the membrane waves, and $\Gamma_{\beta\gamma}^\alpha$ is the Christoffel symbol of the second kind, defined by Eq. (A9). Note that these equations reduce to the equation for geodesics on Σ if c is constant.

The ray-tube area $A(s)$ is the measure of how two neighboring rays spread relative to one another. A differential equation for A follows from Eqs. (8) and (9) by taking variations with respect to a parameter orthogonal to s . The resulting equation is called the variational ray equation or the equation of geodesic variation (when $c \equiv 0$), and is¹⁹

$$c \frac{d}{ds} \frac{1}{c} \frac{dA}{ds} + \left(\frac{1}{R_\parallel R_\perp} + \frac{1}{c} u^\perp{}^\alpha u^\perp{}^\beta \nabla_\alpha \nabla_\beta c \right) A = 0. \quad (10)$$

Thus A is useful for describing the relative spreading or convergence of ray bundles. It is also related to the amplitude of the membrane displacement, $V(s)$, according to the conservation relation

$$\frac{d}{ds} (V\sqrt{mcA}) = 0, \quad (11)$$

valid along a ray. This is also a statement of membrane wave energy conservation in a ray bundle. Equation (11) can be modified to leading order to account for the energy dissipation through acoustic radiation.¹⁷

In this paper we consider solutions of the first-order ray equations (8) and (9), supplemented by Eq. (10) for the ray-tube area. Furthermore, we assume that the membrane wave speed is constant over the surface, so that the set of equations reduces to a pair of coupled first-order differential equations,

$$\frac{d\xi^\alpha}{ds} = u^\alpha, \quad \frac{du^\alpha}{ds} = -\Gamma_{\beta\gamma}^\alpha u^\beta u^\gamma, \quad (12)$$

and

$$\frac{dA}{ds} = cB, \quad c \frac{dB}{ds} = -\frac{A}{R_I R_{II}}. \quad (13)$$

Equations (12) are now the equations for geodesics on the surface Σ . Note that both the ray and ray-tube equations are independent of frequency. This implies that we need only solve these equations once, and can then use the data for the ray trajectories and ray-tube areas at all frequencies considered. This is one of the greatest advantages over other numerical methods where equations have to be solved at each individual frequency.

II. NUMERICAL IMPLEMENTATION

The ray theory discussed above has been verified for fluid-loaded cylindrical and spherical shells.¹⁷ The full strength of the theory, however, is its applicability to smooth shells of arbitrary shape, which has not yet been realized. The main difficulty is the lack of explicit representations for arbitrarily curved surfaces. In most cases a global curvilinear coordinate system cannot be defined on the surface. In the next few subsections, we introduce a comprehensive scheme to obtain explicit representations of arbitrary surfaces, establish local curvilinear coordinate systems and solve the system of ray and transport equations on a shell surface.

A. Local parametric representations of a shell's surface

In principle, one may approximate a closed but arbitrarily curved surface by the positions of a finite set of points, called nodal points, interconnected by a number of curves to form a mesh comprising a finite number of patches. The position of any point on a patch can be approximately described by the position of nodal points and appropriate interpolation functions, called mapping functions. Although it is not feasible to seek an explicit global representation for the geometry of the entire surface, it is always possible, however, to find explicit parametric representations for the geometry of each patch. This procedure originated in the finite element and boundary element methods, and extensive discussions can be found in the texts of Szabó and Babuška,²³ and Zienkiewicz and Taylor.²⁴

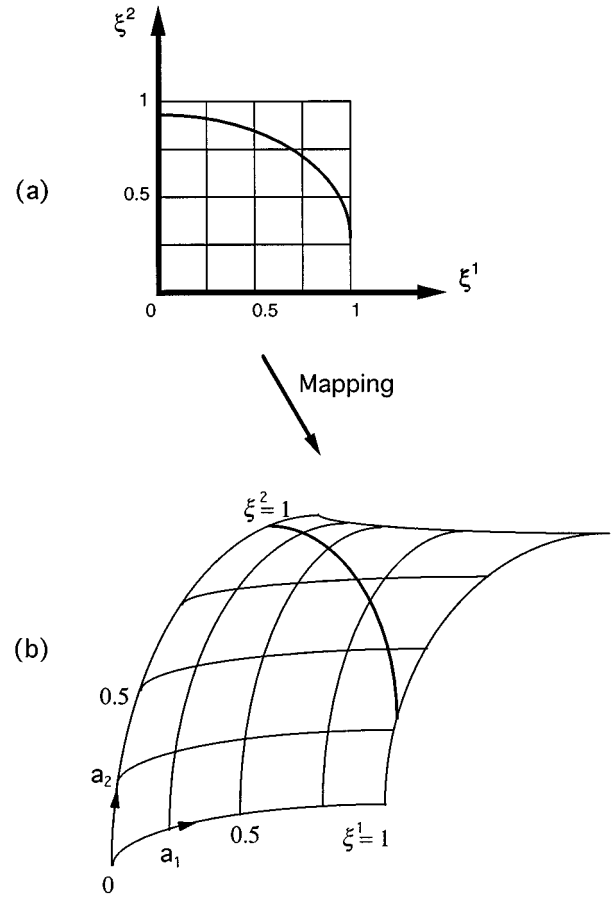


FIG. 1. The mapping from a square patch in the $\xi^1\xi^2$ -plane (a) to a curved patch in (b).

Here, we use quadrilateral patches and introduce the following parametric expression for the position of a point on a patch:

$$x_k^{(N)} = \sum_{i=1}^{p+1} \sum_{j=1}^{p+1} S_{ijk}^{(N)} (\xi^1)^{p+1-i} (\xi^2)^{p+1-j}, \quad 0 \leq \xi^1, \xi^2 \leq 1, \quad (14)$$

where $x_k^{(N)}$ with $k=1,2,3$ are the three covariant components of the position vector in a fixed global Cartesian coordinate system. The parameters ξ^1 and ξ^2 are the local curvilinear coordinates on the patch, and the superscript N denotes the patch number. The order of the polynomial is specified by the integer p . The coefficient matrix in Eq. (14) is uniquely determined by the coordinates of the points on the boundary and interior of the patch. Equation (14) defines the mapping from a flat square shown in Fig. 1(a) to the curved patch in Fig. 1(b). Since the ray theory discussed in Sec. II is valid on a smooth elastic shell, the patches are required to fit together smoothly or approximately smoothly. This condition can be achieved by either decreasing the size of the patches or increasing the order of the polynomial in (14).

When $p=1$ the function (14) maps a square to a flat patch on which the curvature is everywhere zero, and hence provides a poor approximation to a smooth surface and is of no use for our purposes. We therefore restrict the polynomial order p to an integer at least greater than two. In the present

analysis we choose $p=3$ in Eq. (14), which gives a bi-cubic function. It is well known that a cubic function enforces the continuity of curvature along a spatial curve. However, it is not always the case for a 2-D curved surface where a discontinuity in curvature may occur across the boundary between two patches. The sizes of patches must be sufficiently small in order to reduce this discontinuity. We will return to this point later.

B. The local coordinate system

The parameters ξ^1 and ξ^2 may be viewed in a general sense as two curvilinear coordinates within a patch. The local curvilinear coordinate frames at a point in the patch N are then $\mathbf{a}_\alpha$ of Eq. (2) and the normal is $\mathbf{a}_3$ of Eq. (A3). Also, the definitions of the surface metric tensor $a_{\alpha\beta}$, curvature tensor $d_{\alpha\beta}$, and the Christoffel symbols $\Gamma_{\lambda\alpha}^\mu$ remain the same as before (see the Appendix). We note that ξ^1 and ξ^2 are dimensionless parameters, and so the directional vectors along the coordinate frames $\mathbf{a}_1$ and $\mathbf{a}_2$ have dimensions of length. The arc length of a differential line element $d\mathbf{x} = \mathbf{a}_\alpha d\xi^\alpha$ can be evaluated according to

$$ds = (d\xi^\alpha a_{\alpha\beta} d\xi^\beta)^{1/2}. \quad (15)$$

C. Numerical ray tracing

Using the local coordinate system defined above, the system of ray equations assume the same form as given in Sec. I B. For simplicity, we solve the ray Eq. (12) and ray-tube Eq. (13) for the first-order approximation with constant speed c . The analogous equations including higher-order terms can be solved in the same manner.

In performing the numerical integration of these equations we need to enforce two restrictions: that the surface wave normal $\mathbf{u}$ is a unit vector, and that the parameter s is the arc length along a ray path. In other words,

$$u^\alpha a_{\alpha\beta} u^\beta = 1, \quad s = \int_0^s (u^\alpha a_{\alpha\beta} u^\beta)^{1/2} dt. \quad (16)$$

These constraints are satisfied by rewriting the ray equations as

$$\frac{d\xi^\alpha}{ds} = \frac{u^\alpha}{(u^\beta a_{\beta\gamma} u^\gamma)^{1/2}}, \quad (17a)$$

$$\frac{du^\alpha}{ds} = -\frac{\Gamma_{\beta\gamma}^\alpha u^\beta u^\gamma}{u^\beta a_{\beta\gamma} u^\gamma}, \quad (17b)$$

$$\frac{dA}{ds} = \bar{B}, \quad (17c)$$

$$\frac{d\bar{B}}{ds} = -\frac{A}{R_I R_{II}}, \quad (17d)$$

where $\bar{B} = cB$.

The system of ray equations (17) is solved for the initial conditions appropriate to membrane waves launched on a shell,¹⁷

$$\xi^\alpha(0) = \xi^\alpha|_{cp}, \quad u^\alpha(0) = \mathbf{u}^{in} \cdot \mathbf{a}^\alpha|_{cp}, \quad (18a)$$

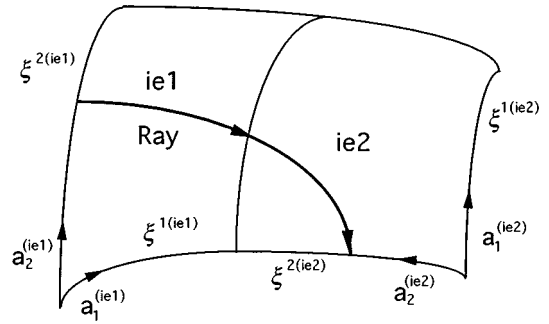


FIG. 2. Schematic illustration of a ray path crossing the boundary between two curved patches labeled by ie1 and ie2.

$$A(0) = 1, \quad \bar{B}(0) = \cot \theta_0 \left. \frac{R_{II}}{R_I R_{II}} \right|_{cp}, \quad (18b)$$

where the symbol " $|_{cp}$ " means that the local coordinates ξ^α and the coordinate frames $\mathbf{a}^\alpha$ are defined on the coupling points. An incident acoustic wave excites the membrane waves only at points on the coupling curves, which are the loci of phase matching according to Snell's law.¹⁷ The coupling points form closed curves on a shell surface with a continuous and smoothly varying surface normal. The scheme to determine the coupling curves is described in the next subsection. The initial value of $A(0)$ is arbitrary, and can be set to unity with no loss in generality. The value of $B(0)$ is derived by matching phase between the incident waves and surface waves at the coupling points.¹⁷

Once the initial conditions (18) are given, it is straightforward to solve the system of ray equations (17) within a single patch. This is accomplished using the fourth-order Runge–Kutta method.²⁵ Special care must be taken when a ray hits the boundary of two adjacent patches, see for example Fig. 2. Then the local curvilinear coordinates ξ^α , and the surface wave normal $\mathbf{u}$ defined in one patch must be transformed into their corresponding values under the new coordinate system defined in the adjacent patch. Assuming the ray is originally traveling in patch N_1 , then hits the boundary separating patches N_1 and N_2 , and enters the patch N_2 , we have

$$\xi^{\alpha(N_2)} = \mathbf{a}^{\alpha(N_2)} \cdot \mathbf{a}_\beta^{(N_1)} \xi^{\beta(N_1)}, \quad (19a)$$

$$u^{\alpha(N_2)} = \mathbf{a}^{\alpha(N_2)} \cdot \mathbf{a}_\beta^{(N_1)} u^{\beta(N_1)}. \quad (19b)$$

The components $\xi^{\alpha(N_2)}$ and $u^{\alpha(N_2)}$ are now defined under the coordinate system of patch N_2 and serve as the initial conditions for tracing the ray in that patch. By following this scheme from patch to patch, we can determine the entire ray trajectory and its associated ray-tube area as it travels over the shell surface.

D. Determining the coupling curve

At the limit of high frequency where the ray theory is applicable, the surface rays are excited by the incident rays only along the coupling curve on the shell's surface where the following condition is satisfied:

$$\mathbf{u}^{in} \cdot \mathbf{a}_3 = -\cos \theta_0. \quad (20)$$

Here θ_0 is the critical angle for the membrane wave (longitudinal or transverse), $\mathbf{u}^{\text{in}}$ is the incident wave propagation direction, and it is assumed that $\mathbf{a}_3$ is directed into the exterior fluid. For example, the critical angle for the excitation of longitudinal waves on the steel shell's surface is 15.82° . The coupling condition (20) means simply that the projection of the incident slowness vector on the shell's surface should equal the membrane wave slowness vector.

The determination of the coupling curves involves seeking the roots of Eq. (20) on the non-Euclidean two-dimensional space, i.e., the curved surface. However, the common difficulties in finding the roots in 2-D spaces do not arise here, because of the fact that the roots of Eq. (20) are not isolated at all but form a closed curve for a given θ_0 . It is reasonable to expect that this closed curve must intersect with some of the curvilinear coordinate curves on the shell's surface. Thus we first mesh the surface Σ into a number of quadrilateral patches and then use a few local curvilinear coordinate curves ξ^1 and ξ^2 to intersect the coupling curve within each patch, see the illustration in Fig. 1(b). In this way the task of finding the root over the curved surface Σ reduces to the problem of root finding along ξ^1 curves and ξ^2 curves, for which many numerical schemes are available.²⁵ Of course, most of ξ^1 curves and ξ^2 curves do not hit the coupling curve. It is also possible that many roots found by this scheme are actually the same. Therefore, in the final step, we sort the roots, eliminate repeated ones and use cubic spline functions to fit them. Thus the coupling curve is represented in a parametric form.

III. NUMERICAL RESULTS AND DISCUSSION

We first discuss the accuracy and efficiency of the present numerical scheme. It is expected that the numerical errors originate primarily from two sources: The first is the approximation of the shell's surface, namely the discretization and parametric representation of the geometry of each patch, and the other is introduced by the numerical integration of the ray equations (17). These two sources of errors, in principle, disappear as the sizes of the patches and the integration step become sufficiently small. The two errors are independent of each other, and can therefore be assessed separately. In the following subsections we examine the effect of each error by varying the size of the patches on a spherical shell while leaving the integration step fixed or *vice versa*. We subsequently apply the scheme to some examples involving nonseparable geometry.

A. Tests of the accuracy of the numerical scheme

1. The effect of the patch size

A spherical thin elastic shell of radius a is first meshed into a number of patches approximately equal in size. We consider three cases: the same shell of 24 patches, 96 patches, and 384 patches, while fixing the integration step Δs at $0.002a$. This integration step is small enough to guarantee the convergence of the numerical integration. In this way we can see clearly how patch sizes affect the accuracy of our numerical results on the coupling curves, ray trajectories and ray-tube areas.

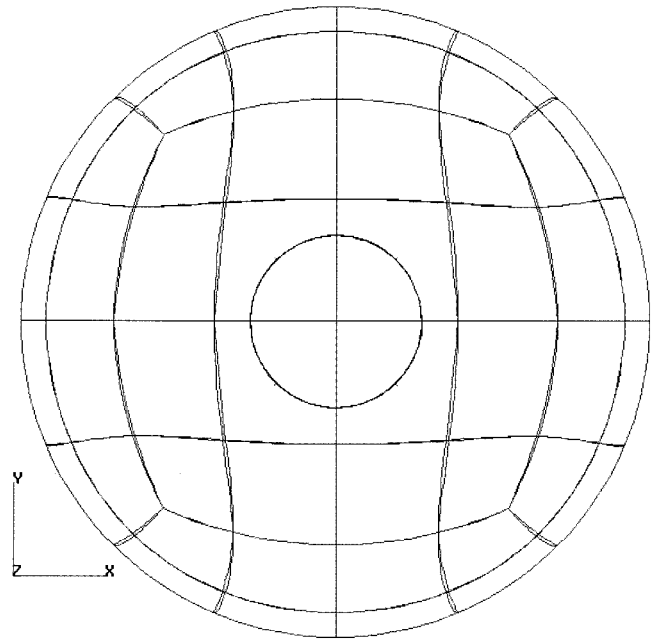


FIG. 3. Two coupling curves computed respectively from the same sphere of 24 patches and 96 patches.

The coupling curves computed respectively from the same spherical shell of 24 patches and 96 patches are plotted in Fig. 3. The difference between the two curves is negligible. This not surprising when one considers that the coupling condition (20) depends only on the position vectors and the normals of all the points on the shell. The overall shape of the shell as represented by 24 patches or by 96 patches is virtually the same except at the corners of some patches. The agreement in Fig. 3 therefore confirms that the 24-patch shell provides a good approximation up to the first derivative on the shell.

Once the coupling curve is determined, the initial values for solving the ray system equations (17) are given by Eqs. (18). Five ray paths, generated at five coupling points that are approximately equally spaced along the coupling curve, are presented in Fig. 4. The results are computed on the same shell with three different number of patches, i.e., 24, 96, and 384. The fact that each ray trajectory on a sphere is a great circle provides a criterion to judge the accuracy of the present numerical results. We notice from Fig. 4(a) that some of the ray paths on the 24-patch shell deviate slightly from the exact solution. This deviation disappears in Fig. 4(b) and (c) when the number of patches increases from 24 to 96 or 384, and all the ray paths converge to the exact ray paths.

The evolution of the ray-tube areas along the five ray paths appearing in Fig. 4 are shown in Fig. 5. The exact solution, marked by the squares in Fig. 5, is also given as a reference. The results of the 24-patch shell in Fig. 5(a) are quite dispersive and exhibit considerable deviation from the exact one. But all the discrepancies are reduced noticeably for the 96-patch shell and are almost completely eliminated for 384 patches. To further understand this phenomenon, we examined the numerical error in the principal radii of curvature on the shell's surface. It turns out that the principal radii at some points along the rays on the 24-patch shell could

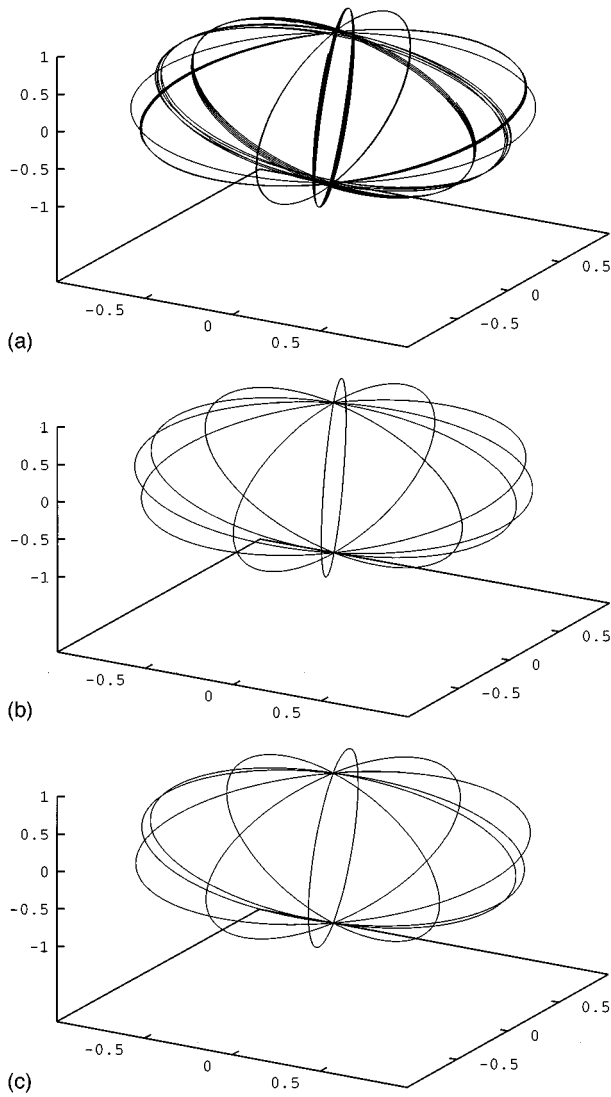


FIG. 4. Five ray trajectories on the sphere of (a) 24 patches, (b) 96 patches, and (c) 384 patches.

have as much as 10% error. These errors reduce to less than 5% on the 96-patch shell and 1% on the 384-patch shell. We note that the evolution equation for the ray-tube area depends explicitly on the Gaussian curvature. Thus, accurate evaluation of the curvature tensor is critical to the computation of ray-tube areas.

The amplitude of the membrane wave field, $V(s)$, is related to the ray-tube area by Eq. (11), or $V \propto A^{-1/2}$. Since the phase of the membrane wave is ks , where $k \approx \omega/c$ is constant up to the first order along a ray, the phase varies linearly with the arc length. Thus the abscissa in Fig. 5 actually gives the phase while the ordinate yields information about the wave amplitude. Figure 5 provides a comparison of the exact result with the present numerical results computed for the shells with three different patch numbers. Evidently, the numerical results converge uniformly as the patch number increases from 24 to 96 and then 384.

Since the position vector of a point is interpolated here by the bi-cubic function (14), the curvature tensor, which depends on the second derivatives of the position vector, is interpolated by a linear function. Obviously, a linear interpo-

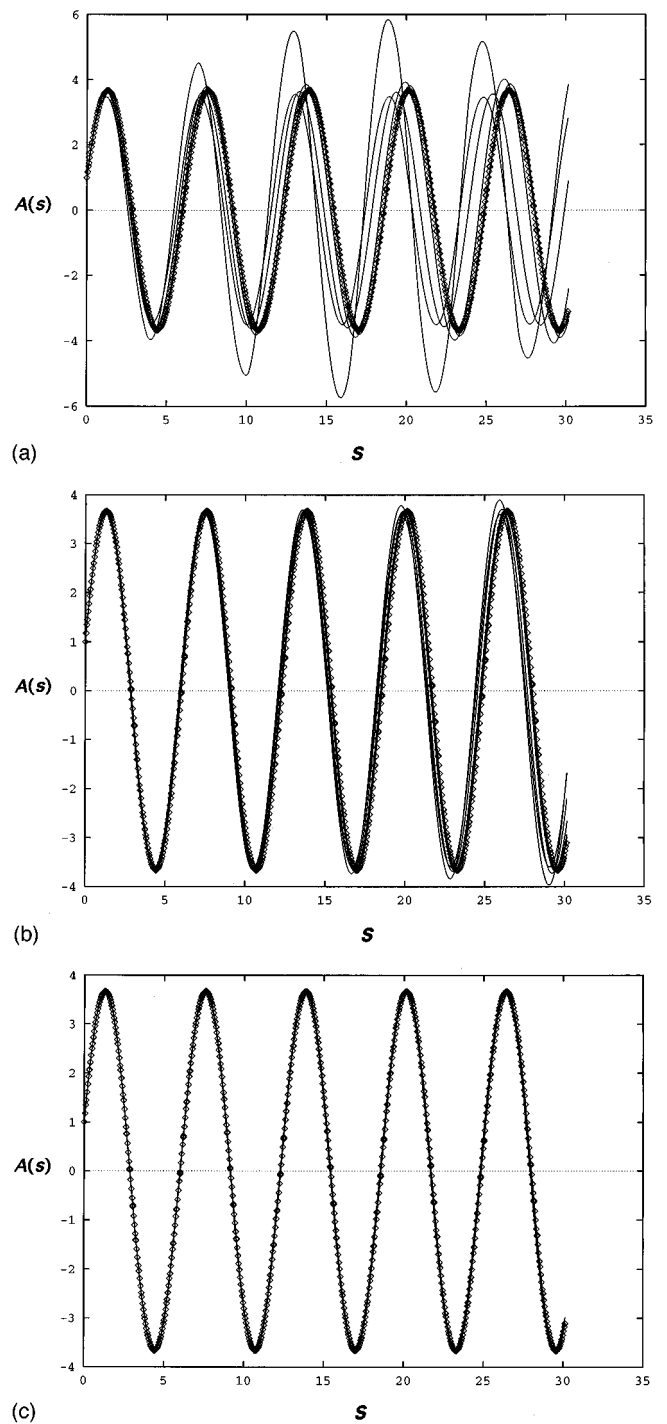


FIG. 5. The evolution of the ray-tube areas associated with the five ray paths given by Fig. 4 versus the arc length along each ray: (a) 24-patch sphere, (b) 96-patch sphere, and (c) 384-patch sphere. The numerical results are represented by the solid lines while the exact result is marked by the small squares.

lation provides a good approximation only when the patch size is sufficiently small. In order to use a small number of patches or a large patch size, it is necessary to consider higher-order polynomials as interpolation functions in Eq. (14). In fact, increasing the order of the polynomial does not introduce any new complexities or require more computational time. But it may present some inconvenience in the determination of the coefficients in Eq. (14) from the posi-

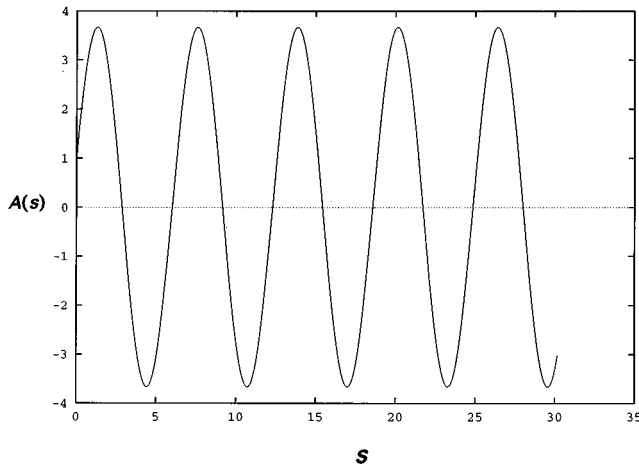


FIG. 6. Comparison of exact and numerical results of the ray-tube area along a ray trajectory on the sphere of 384 patches. The numerical results are computed respectively at the integration steps $h=0.002a$, $0.02a$, and $0.05a$, where a is the sphere radius.

tions of the nodal points. This problem has been solved using standard procedures as described in the text of Szabó and Babuška.²³

2. The effect of integration step size

To examine numerical errors introduced by inappropriate integration steps, we vary the integration step gradually while keeping the patch number of the spherical shell fixed at 384. We saw above that the error caused by the discretization of the shell's surface is negligible for this patch number. Thus any visible deviation from the exact solution should indicate an error due to the numerical integration. In the present scheme the integration step h is not allowed to exceed the smallest dimension of the patches and h is automatically decreased when a ray approaches a patch edge. The average dimension of the patches is about $0.2a$ for the 384-patch shell, where a is the radius of the sphere.

We choose the largest integration step h equal to $0.002a$, $0.02a$, and $0.05a$, respectively, and compare the numerical results with the exact one in Fig. 6. The deviation is almost

imperceptible. Using the present scheme we are not able to check the effect of larger h because the size of integration is limited by the patch size. In order to see appreciable effect of integration step, one needs to use large patches with higher-order polynomial as interpolation function. The results in Fig. 6 suggest that $h=0.05a$ is sufficiently accurate for rays on the 384-patch spherical shell.

B. Applications

The ray-based numerical scheme is next applied to two examples involving nonseparable geometry. We are particularly interested in the effect of variable curvature on membrane wave propagation. The curvature governs the ray paths and the ray-tube amplitudes, and is therefore expected to make the rays behave as if they were propagating on a flat surface that is both inhomogeneous and anisotropic.

1. Example A

The first example considered here is an idealized submarine model as shown in Fig. 7. It is composed of two hemispherical endcaps of unit radius and a cylindrical shell eight units long with a smooth hump. The shape of the cylindrical section is described by its cylindrical coordinates (r, z, ϕ) such that

$$r = 1 + 1.8 \sin^{12}[\pi(z+1)/8] \sin^{16}(0.5\phi),$$

$$0 \leq z \leq 8, \quad 0 \leq \phi \leq 2\pi. \quad (21)$$

The whole shell is divided into 352 patches. The incident plane acoustic wave is directed perpendicularly inward to the plane of Fig. 7. Without the hump, all the rays initiating at the coupling points along the cylinder would travel circumferentially as plane circles. When the smooth hump is introduced, most of the rays originating at the coupling points near the hump deviate away from plane circles and travel all over the shell's surface, see, for example, the trajectory displayed in Fig. 7. Reflections at the junctions would generate new rays, but these are not shown here. Since the hump destroys the regular helical or circular paths of the rays on a

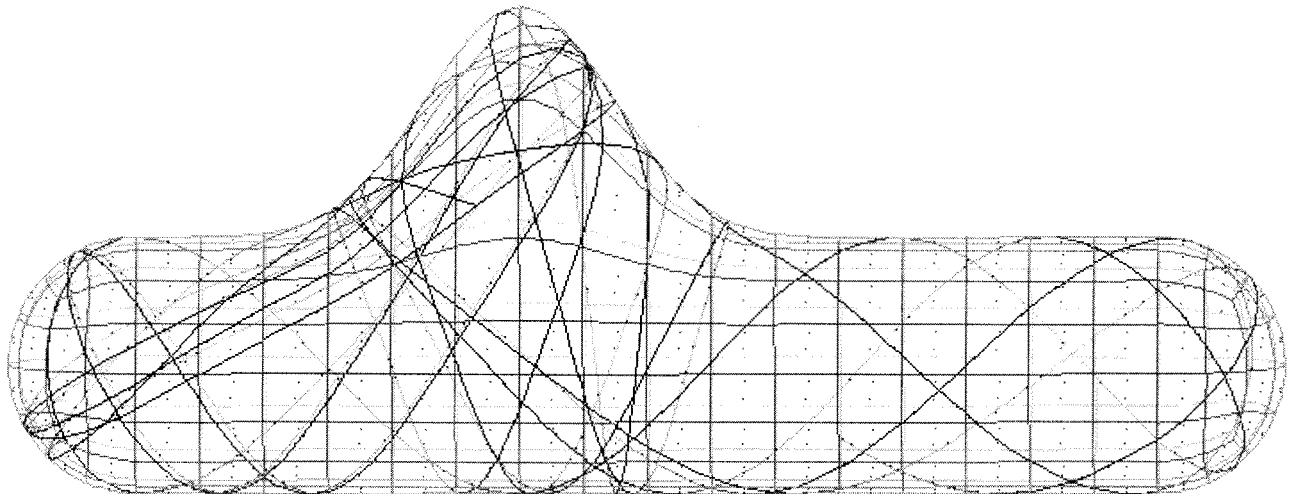


FIG. 7. A ray trajectory on an idealized submarine. The incident direction points inward to the plane of this figure.

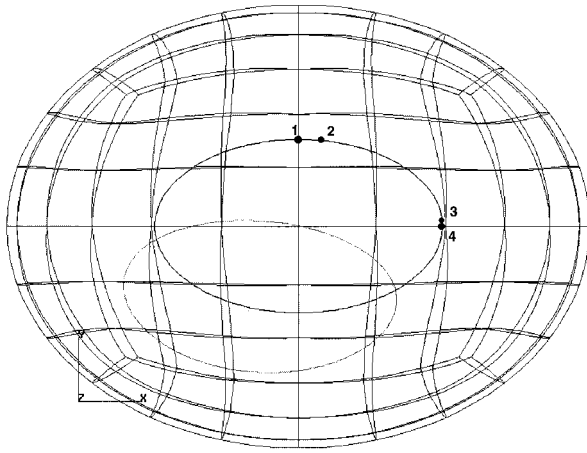


FIG. 8. The coupling curves for longitudinal waves on an ellipsoidal shell subject to an acoustic plane wave at the incident directions: $\theta=180^\circ$, $\phi=0^\circ$ (the curve symmetric about Z axis), and $\theta=180^\circ$, $\phi=70^\circ$ (the curve asymmetric about Z axis).

cylinder, conceivably it will significantly alter the resonant pattern of the shell.

2. Example B

We now consider a thin elastic shell of ellipsoidal shape shown in Fig. 8, with the ratio of its axes as $a:b:c=2:1.5:1$. The ellipsoid is obtained by uniformly stretching a unit sphere along its X and Y axes by a factor of 2 and 1.5, respectively. In order to minimize the numerical errors, we choose the patch number as 384 and the integration step as 0.002. The coupling longitudinal curves excited by the two different incident directions are specified by their spherical coordinates and are plotted in Fig. 8. The one symmetric about Z axis corresponds to the incident direction at $(\theta, \phi) = (180^\circ, 0)$ whereas the asymmetric one is generated by the incident wave along the direction at $(\theta, \phi) = (164.18^\circ, 70^\circ)$. Four coupling points for the incidence $(\theta, \phi) = (180^\circ, 0)$ are selected and labeled by 1 through 4. The precise positions of these points are given by their Cartesian coordinates: $\mathbf{r}_1 = (0.000, 0.587, 0.920)$, $\mathbf{r}_2 = (0.184, 0.577, 0.919)$, $\mathbf{r}_3 = (0.986, 0.009, 0.870)$, $\mathbf{r}_4 = (0.986, 0.000, 0.870)$. A little bit of calculation shows that the coupling curve is not an exact plane curve on the ellipsoid, as it appears to be in Fig. 8.

Four ray paths initiating at these four coupling points are presented in Fig. 9. The rays are labeled by the indices of their respective coupling points in Fig. 8. Two rays originating at coupling point 1 and 4 are shown in Fig. 9(a) where they remain in the YZ plane and XZ plane as closed orbits. The stability of these two closed orbits is illustrated qualitatively in Fig. 9(b) and (c) by slightly shifting their initial positions from coupling points 1 and 4 to the points 2 and 3. It is interesting to note that ray 2 stays in the vicinity of ray 1 along its entire trajectory, while ray 3 deviates substantially from ray 4. This suggests that ray 1 is a stable path while ray 4 is not.

The evolution of ray-tube areas along these four ray paths are shown in Fig. 10(a) and (b). These provide a quantitative illustration of ray spreading and stability along different rays. The maximum amplitude of the ray-tube area

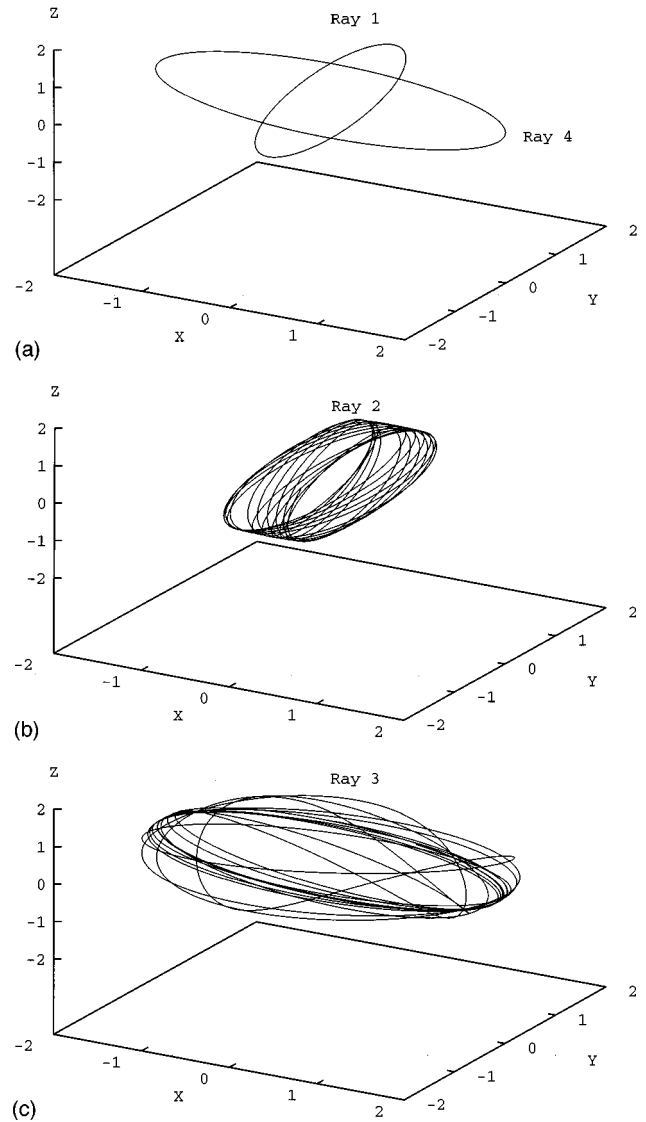


FIG. 9. Ray trajectories of arc length equal to 125 originating at the coupling points illustrated in Fig. 8. (a) Two closed ray paths start at the coupling points 1 and 4, respectively. (b) The ray originates at the coupling point 2. (c) The ray originates at the coupling point 3.

along a given ray increases uniformly as the coupling point moves from 1 to 4, and it becomes unbounded along ray 4. The growth of the ray-tube area is indicative of the stability of its ray trajectory: the larger the maximum of the ray-tube area, the less stable the ray path. Intuitively, the magnitude of a ray-tube area is proportional to the length of wavefront between two neighboring rays traveling on a curved shell's surface. A large value for the ray-tube area indicates that a small deviation from the initial position and direction of a ray may lead to a completely different ray trajectory. In this sense, the maximum of the ray-tube areas along a ray may be used to describe the stability of the ray trajectory.

Finally, we examine caustic formation in Fig. 11. The first caustic is defined as the locus of the first points on the rays at which the ray-tube area shrinks to zero. Thus Fig. 11(a) shows the first such points for 15 rays which are equally spaced on the coupling curve, and the associated ray-tube area of each ray is displayed in Fig. 11(b). The first

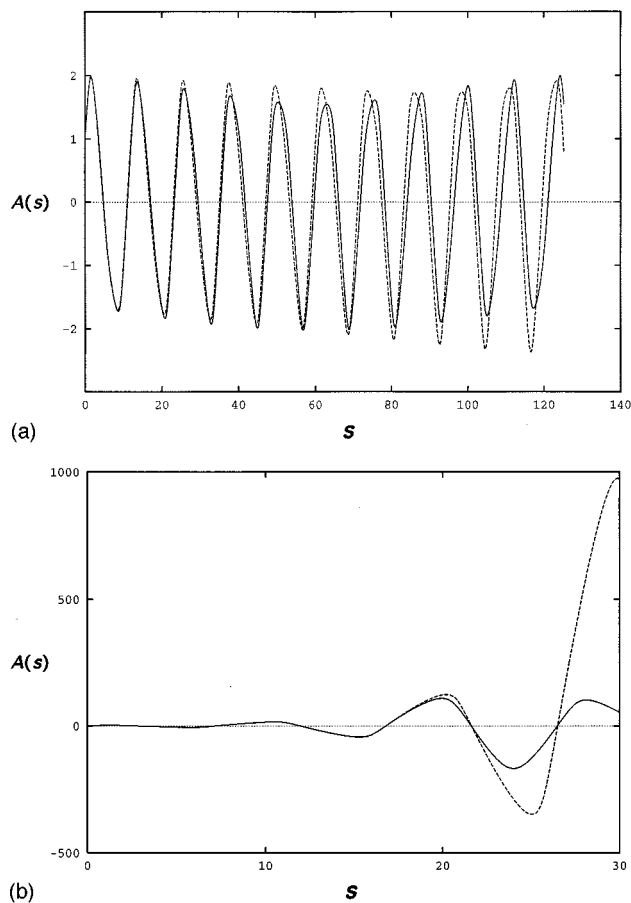


FIG. 10. The evolution of the ray-tube areas along the ray trajectories shown in Fig. 9. In (a), the solid line is for ray 1, the dashed line for ray 2. In (b) the solid and dashed lines are for rays 3 and 4, respectively.

caustic curve is spread over a finite range on the ellipsoidal shell, unlike the case of a spherical shell where the caustic degenerates to a focal point. As we trace the rays further the caustics eventually become distributed over the entire shell surface. This presents some difficulties in the synthesis of the surface field and the scattered acoustic response because the amplitude of a ray at a caustic is singular. This problem can be solved by extending the ray theory into complex space through the use of Gaussian beams.²⁶ As the consequence, the amplitude remains finite at a caustic.

IV. CONCLUSION

We have developed a ray-based numerical scheme to describe elastic wave propagation on a fluid-loaded, smooth thin elastic shell of arbitrary shape. The method is based on the accurate representation of a shell's surface by a number of patches, each of which yields explicit forms for the parameters entering the ray equations, such as Gaussian surface curvature. The numerical scheme as developed here fully realizes the strength of the ray theory and has several advantages for applications. First, as is common in ray methods, the present scheme directly solves a set of ordinary differential equations along ray paths, rather than solving the original partial differential equations on a 2-D curved surface. This makes the scheme much more efficient and accurate at high frequencies as compared with other numerical methods, such

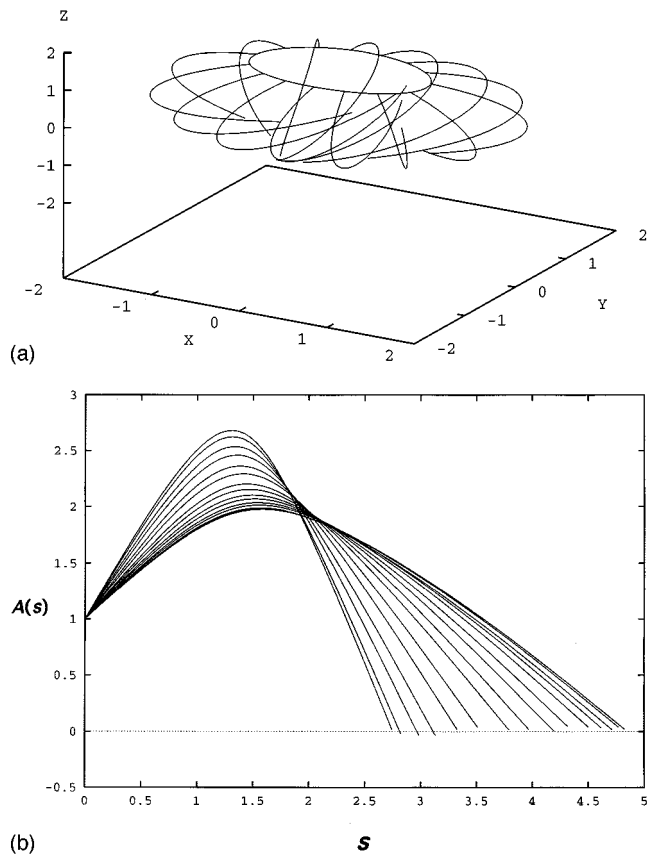


FIG. 11. (a) Fifteen rays starting at 15 coupling points equally spaced along the coupling curve and ending at their first caustics. (b) The ray-tube areas along the 15 rays.

as finite element and finite difference methods. The present scheme has the ability and flexibility for handling complicated configurations because most of the numerical computations are carried out at the local level of patches. Finally, most of the computed numerical quantities such as ray paths, ray-tube areas and the number of patches used, are independent of frequency. This is another important advantage over the finite element and boundary element methods since the number of elements in these approaches has to increase substantially with the frequency in order to achieve reasonable accuracy.

The accuracy of the present numerical results depend primarily on the choice of patch size and the integration step. The test cases considered show that the errors associated with these parameters are negligible as the patch sizes approach $0.2R_{\min}$ and integration step reduces to $0.05R_{\min}$. These restrictions might be loosened a little if higher-order polynomials are used as interpolation functions within each patch.

The inhomogeneous distribution of curvature on a shell's surfaces has significant effects on elastic wave propagation. Thus the spreading of ray trajectories exhibits strong directional or anisotropic dependence, which leads to complicated caustic patterns. Ray trajectories can also lose their stability. On an ellipsoidal shell, for example, rays originating at the coupling points of higher Gaussian curvature are less stable than those starting at the points of lower Gaussian curvature, and caustics appear almost everywhere on the sur-

face if the rays are traced long enough. Our numerical results show that the evolution of the ray-tube area along a ray provides a good quantitative description of the spreading and stability of the ray. Unstable ray paths always correspond to very large or unbounded maximums for the ray-tube areas.

The scheme presented here is useful in revealing the physical mechanisms of elastic wave propagation on a fluid loaded shell. Most importantly, it will serve as a framework for the construction of the surface fields and the scattered fields.²⁶

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APPENDIX: LOCAL GEOMETRY OF A SHELL'S SURFACE

The surface metric tensor $a_{\alpha\beta}$ and the curvature tensor $d_{\alpha\beta}$ are symmetric tensors, defined at a point on Σ by

$$a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta \quad (\text{A1})$$

and

$$d_{\alpha\beta} = -\mathbf{a}_3 \cdot \mathbf{a}_{\alpha,\beta}, \quad (\text{A2})$$

respectively, where $\mathbf{a}_3$ is the unit normal to Σ ,

$$\mathbf{a}_3 = |\mathbf{a}_1 \times \mathbf{a}_2|^{-1} \mathbf{a}_1 \times \mathbf{a}_2. \quad (\text{A3})$$

The contravariant forms of the three vectors $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ are

$$\mathbf{a}^\alpha = a^{\alpha\beta} \mathbf{a}_\beta, \quad \mathbf{a}^3 = \mathbf{a}_3,$$

where $a^{\alpha\beta}$ is the contravariant form of the surface metric tensor $a_{\alpha\beta}$, defined such that $a^{\alpha\gamma} a_{\gamma\beta} = \delta^\alpha_\beta$, the identity tensor. The mixed form of the curvature tensor, d^α_β , is related to its covariant form by a linear transformation

$$d^\alpha_\beta = a^{\alpha\gamma} d_{\gamma\beta}. \quad (\text{A4})$$

The two principal radii of curvature, R_I and R_{II} , are the inverse of the eigenvalues of the curvature tensor, or

$$d^\alpha_\beta Q_I^\beta = \frac{1}{R_I} Q_I^\alpha, \quad d^\alpha_\beta Q_{II}^\beta = \frac{1}{R_{II}} Q_{II}^\alpha, \quad (\text{A5})$$

and Q_I^α and Q_{II}^α define the principal directions of curvature. The product of the two principal curvatures, the Gaussian curvature, follows as

$$\frac{1}{R_I R_{II}} = \frac{\det(d_{\alpha\beta})}{\det(a_{\gamma\lambda})}. \quad (\text{A6})$$

The derivatives of a vector $\mathbf{v}$ with respect to the curvilinear coordinates (ξ^1, ξ^2) are given by²¹

$$\mathbf{v}_{,\alpha} = (\nabla_\alpha v_\lambda + d_{\alpha\lambda} v_3) \mathbf{a}^\lambda + (v_{3,\alpha} - d^\lambda_\alpha v_\lambda) \mathbf{a}_3, \quad (\text{A7})$$

where

$$\nabla_\alpha v_\lambda = v_{\lambda,\alpha} - \Gamma^\mu_{\lambda\alpha} v_\mu, \quad (\text{A8})$$

and $\Gamma^\mu_{\lambda\alpha}$ are the Christoffel symbol of the second kind, defined by

$$\Gamma^\mu_{\lambda\alpha} = a^{\mu\gamma} \mathbf{a}_\gamma \cdot \mathbf{a}_{\lambda,\alpha}. \quad (\text{A9})$$

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