

Benchmarking an acoustic coupling theory for elastic shells of arbitrary shape

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Coupling coefficients for thin shells of arbitrary curvature based upon a ray-theoretic approach devised by the authors are benchmarked against exact and high-frequency results for canonical geometries, i.e., the spherical and infinite circular cylindrical shells. The well-known high-frequency approximations of the coupling coefficients are found to describe their magnitude accurately but not the phase for midfrequencies (between ring and coincidence) when compared with results obtained by applying the Sommerfeld–Watson transformation to the exact modal series. The coupling coefficients characterize both the magnitude and the phase accurately and simply in terms of fundamental physical parameters. © 1995 Acoustical Society of America.

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INTRODUCTION

Acoustic scattering from thin elastic shells can be determined exactly for two canonical geometries only, i.e., circular cylindrical and spherical shells,¹ using normal-mode expansions. Other methods must be considered for other geometries. We have recently developed a ray theory description of scattering from a shell of arbitrary shape² in which the coupling mechanism for acoustic-to-membrane waves was derived analytically. Here, membrane means waves of longitudinal or shear nature, as distinct from flexural waves. The formulation is entirely in physical space and does not require transforms or spectral integrals. These ray-based coupling coefficients are explicitly determined by physical quantities which describe the thin elastic shell and the surrounding fluid. Knowledge of the launching and detachment coefficients at any point on the shell's surface allows one to determine the scattered pressure wave field shed into the surrounding fluid for any smooth convex shell. We have compared far-field form function predictions of the asymptotic theory with the normal-mode series for spherical and cylindrical shells.² The accuracy was remarkable considering the simplicity of the approximate coupling coefficients.

Here we benchmark the asymptotic theory by comparing one of its central elements—the coupling coefficient—with an “exact” analog. The exact solution is chosen to be the normal-mode series for thin elastic shells of cylindrical or spherical shape. A ray-type representation is then obtained by unraveling the modal description using either the Poisson summation identity or the Watson transformation,^{3–6} both of which are applicable only to these canonical geometries. This form of the solution clearly exhibits the rays reflecting specularly from the shell and those which travel over its surface.^{4–7} Terms involving the leaky surface rays require the evaluation of a residue containing Hankel functions of complex order.^{5,8} This factor has been approximated by several authors in order to decrease the difficulty of calculating the ray-based solution.^{8–10} There have also been several attempts to modify the canonical results so that they may be

extended to shells of arbitrary shape using empirical arguments.⁹ In contrast, the authors' coupling theory for shells of arbitrary curvature² stands on firm analytical ground and has been implemented by Yang *et al.*¹¹

We begin in Sec. I with the definition of the coupling coefficient. The ray-theoretic form of the coupling coefficient, based on the theory of Norris and Rebinsky² for a shell of arbitrary shape, is described in Sec. I A. The coupling coefficient obtained from the normal modal series by application of the Sommerfeld–Watson transformation is described in Sec. I B 1, and the related high-frequency approximation is discussed in Sec. I B 2. The results of numerical experiments are presented in Sec. II, where the exact solution, the ray-theoretic method, and the high-frequency approximation are compared for circular cylindrical and spherical thin elastic shells.

I. COUPLING COEFFICIENT

A time harmonic acoustic plane wave with pressure p^{inc} is incident on a smooth closed shell and the scattered pressure is characterized by the form function

$$\mathcal{F} = \lim_{r \rightarrow \infty} [(2r/a)^{(d-1)/2} e^{-ik_f r} (p - p^{\text{inc}})], \quad (1)$$

where p is the total pressure wave field, d is the spatial dimension, i.e., $d=2$ for the cylinder and $d=3$ for the sphere, and a is a reference length which will be taken as the target radius. Also $k_f = \omega/c_f$, and $e^{-i\omega t}$ is understood but suppressed. The form function may be split into two distinct parts,^{1,3,12}

$$\mathcal{F} = \mathcal{F}_{\text{spec}} + \mathcal{F}_l, \quad (2)$$

where $\mathcal{F}_{\text{spec}}$ is the specular contribution and $\mathcal{F}_l$ represents the leaky surface waves which traverse the shell. This separation is physically meaningful only in the asymptotic limit of high frequency, $k_f a \gg 1$. The asymptotic description is well documented for cylindrical or spherical shells, for which the form function can be represented in modal series form. A geometrical optics, or ray-type solution is obtained

by applying the Poisson summation formula or the Watson transformation, yielding explicit spectral integral contributions for both the specular and leaky surface wave components.^{1,3,5,6}

The cylinder and sphere are extremely degenerate in that all membrane waves which radiate to the far field are of the same form, modulo the phase factor for one circumnavigation: $e^{i2\pi ka}$, where k is the longitudinal membrane wave number. The repetitive nature of the surface rays permits one to combine all membrane effects in a single term. For instance, for backscatter,

$$\mathcal{F}_l^{c,s} = -G^{c,s} e^{-i2k_f a \cos \theta_0} \frac{e^{i2ka(\pi - \theta_c)}}{1 + j e^{i2\pi k a}} e^{-i(j+1)\pi/4}, \quad (3)$$

where c and $j = -1$ refer to a cylindrical shell, s and $j = 1$ to a spherical shell. Also, $ka \approx k_p a + i\beta^{c,s}$ where $k_p = \omega/c_p$ is the wave number in the flat plate limit, and $\theta_0 = \sin^{-1}(k_p/k_f)$ defines the critical angle of the longitudinal surface wave.

Our main concern in this letter is with the “coupling coefficients” G^c and G^s appearing in Eq. (3) (this notation is consistent with Marston⁵). The coupling coefficient characterizes the degree of acoustic-to-membrane and membrane-to-acoustic wave interaction. More generally, one could define separate coefficients for coupling onto and off of a shell. Thus the former determines the amplitude of the in-plane displacement generated when a supersonic membrane wave is launched on an arbitrary shaped thin elastic shell.² Defining the coupling coefficients for attachment and detachment in terms of the surface pressure allows one to easily evaluate scattering from shells of varying curvature¹¹ but here we use the traditional definition based upon the form function for ease of the reader. Thus we estimate the quantities G^c and G^s in terms of these more elemental coefficients following the general procedure outlined in Ref. 2. The purpose of this Letter is to compare and benchmark these predictions with the estimates for G^c and G^s from the “exact” spectral representations.

The determination of G^c and G^s from Eqs. (2) and (3) requires distinguishing the specular and leaky membrane wave contributions. Each term is characterized by a simple amplitude and phase, as discussed above. Any attempt at assessing the accuracy of G^c and G^s from a particular model inevitably depends upon our ability to first subtract the specular term, which is only defined in an asymptotic sense. This limitation should be borne in mind, because it means that the definition of the coupling coefficient depends on what we take for the specular response.¹³ In general, the leading-order specular field depends upon the reflection coefficient for a flat plate, but higher-order approximations can be obtained. For instance, Ho¹⁴ derived a correction to the leading-order reflected wave field by using Debye expansions for the Hankel functions appearing in the modal series. The specular field used in the ray theory formulation below is determined from an impedance boundary condition for the total pressure.² Higher-order terms in the specular response from curved impedance surfaces could be obtained using the procedure outlined by Keller *et al.*,¹⁶ but we do not pursue this here.

A. Pure ray theory for arbitrary shell curvature

A quantitative theory that describes how membrane waves are launched on and shed from arbitrarily shaped elastic shells subject to heavy fluid loading was recently derived by the authors.² The method employs ray theory to represent the supersonic membrane waves traveling along the shell. Their excitation occurs when the ray transport equation is forced by the phase-matched incident wave field, and the coupled wave amplitude for the in-plane displacement is obtained by solving this ordinary differential equation [cf. Eqs. (68)–(76) of Ref. 2]. The scattered or radiated pressure can also be calculated [cf. Eqs. (77)–(84) of Ref. 2] and hence the form function for acoustic scattering from an elastic shell of any smooth convex shape may be determined.^{2,11} By simplifying these expressions to the cylindrical and spherical geometries, closed-form expressions for the form functions can be determined from the general formalism, and we refer to Norris and Rebinsky² for details. In terms of the present notation, we have

$$G^c = 4\pi B / \sqrt{i\pi k_f a}, \quad G^s = -4\pi i \sin \theta_0 B, \quad (4)$$

with

$$B = \frac{1}{k_p a} \frac{Z_m Z_s}{(Z_m + Z_s)^2} \left(\frac{a}{R_0} \right)^2 \quad (5)$$

and

$$Z_m = -i\omega\rho h, \quad Z_s = \rho_f c_f \sec \theta_0, \quad (6)$$

where ρ_f , ρ are the fluid and shell densities, respectively, and h is the thickness. The coupling coefficients of Eq. (4) are remarkably simple, and contain three physical quantities: the shell’s inertial impedance Z_m , the fluid impedance Z_s , and an effective radius of curvature, R_0 . The general theory of Ref. 2 includes both longitudinal and shear wave effects, but only longitudinal waves are excited on the spherical shell and on the cylindrical shell under broadside insonification. The effective radius of curvature for longitudinal waves is $R_0 = (R_{\parallel}^{-1} + \nu R_{\perp}^{-1})^{-1}$, where $R_{\parallel}$ and $R_{\perp}$ are the radii of curvature along and normal to the ray path on the shell and ν is Poisson’s ratio. Thus $R_{\parallel} = a$ and $1/R_{\perp} = 0$ for a cylindrical shell, and $R_{\parallel} = R_{\perp} = a$ for a spherical shell.

The solution to the dispersion relation for the longitudinal wave number, Eq. (14), may be approximated by^{2,18}

$$(ka)^2 \approx (k_p a)^2 + \frac{1 - \nu}{R_{\parallel} R_{\perp}} - \frac{Z_m}{Z_m + Z_s} \left(\frac{a}{R_0} \right)^2, \quad (7)$$

where $R_{\parallel}$ and $R_{\perp}$ are the principal radii of curvature. This approximation for ka provides a simple method to calculate the radiation damping and phase-shift factor of Eq. (3).

The ratio of the impedances in Eq. (6) can be written $Z_s/Z_m = i\Omega_0/\Omega$, where $\Omega = k_f a \cos \theta_0$ and

$$\Omega_0 = \rho_f a / \rho h, \quad (8)$$

is a dimensionless null frequency which demarcates the transition between low-, $\Omega < \Omega_0$, and high-, $\Omega > \Omega_0$, frequency regimes which are governed by pressure release or rigid behavior, respectively. In the limit $\Omega \gg \Omega_0$, Eqs. (7) and (5) imply that

$$\beta^{c,s} \approx \frac{1}{2k_p a} \frac{\Omega_0}{\Omega} \left(\frac{a}{R_0} \right)^2, \quad B \rightarrow 2i\beta^{c,s}. \quad (9)$$

High-frequency versions of the coupling coefficients then follow from Eqs. (4) and (9) as

$$G_\infty^c \approx e^{i\pi/4} 8\pi\beta^c / \sqrt{\pi k_f a}, \quad G_\infty^s \approx 8\pi\beta^s \sin \theta_0. \quad (10)$$

We will refer to the approximations G_∞^c and G_∞^s of Eq. (10) as the “high-frequency” coupling coefficients.

B. Review of thin shell theory

1. Normal-mode series

The ray representation of Eq. (3) can also be obtained from the exact wave series solution by application of the Sommerfeld–Watson transformation.^{5–7} We consider only backscatter from thin elastic cylindrical and spherical shells, for which the exact coupling coefficients are

$$G^c = 4\pi A^c / \sqrt{i\pi k_f a}, \quad G^s = -4\pi i \sin \theta_0 A^s, \quad (11)$$

where

$$A^{c,s} = D^{c,s} (Z_d^{c,s} - Z_1^{c,s}) \left(\frac{\partial}{\partial m} (Z_d^{c,s} - Z_1^{c,s}) \right)^{-1} \bigg|_{m=m_p} \quad (12)$$

and the complex number m_p is the root of the dispersion relation $Z_d^{c,s} - Z_1^{c,s} = 0$. The impedances $Z_{1,2}^c = (\Omega_0/k_f a) \times [H_m^{(1,2)}(k_f a)/H_m^{(1,2)'}(k_f a)]$, and phase parameter $D^c = i\Phi H_m^{(2)'}(k_f a)/H_m^{(1)'}(k_f a)$, where $\Phi = e^{i2k_f a \cos \theta_0} \times e^{-ika(\pi - 2\theta_0)}$. By replacing the Hankel functions of the first and second kind $H_m^{(1,2)}$ with their spherical counterparts $h_m^{(1,2)}$, $Z_{1,2}^s$, and D^s are obtained. The membrane wave number $ka = m_p$, $m_p + 1/2$ for the circular cylindrical shell and spherical shell, respectively. The ' indicates differentiation with respect to the argument. The structural impedances $Z_d^{c,s}$ depend upon the particular thin shell theory adopted,^{7,8,11,13,14} however, it turns out that the differences between them are insignificant for the cases considered here. Thus we have chosen the structural representation used by Norris and Vasudevan.¹³

2. High frequency

This limit applies when the frequency of the incident wave field is much greater than the null frequency of the elastic shell, or $k_f a \gg \Omega_0$. Evaluation of the coupling coefficients in Eq. (11) then requires knowledge of how $A^{c,s}$ and $D^{c,s}$ behave as $k_f a \rightarrow \infty$. The Hankel functions in $A^{c,s}$ and $D^{c,s}$ can be suitably approximated, giving $A^{c,s} \rightarrow 2i\beta^{c,s}$ and $D^{c,s} \rightarrow 1$.^{5,8,10} A detailed derivation is given in Appendix E of Ho and Felsen.¹⁰ Substitution of these limits into Eqs. (11) yields the same high-frequency estimates of the coupling coefficients as in Eq. (10). Similar results have been obtained by Ho and Felsen¹⁰ for the spherical shell and Felsen *et al.*⁸ for the cylindrical shell.

Analogous coefficients for elastic cylinders and spheres, either shells or solids, were obtained by Marston⁵ using resonance scattering theory (RST). Marston^{3,5} subsequently applied these approximations to estimate the coupling coefficients for elastic spherical and cylindrical shells and obtained

good agreement with the exact solution for frequencies beyond coincidence.³ Ho and Felsen¹⁰ also note that this is an accurate approximation if the elastic shell is sufficiently thick. But, for thin elastic shells insonified in the midfrequency region, defined as that between the ring and coincidence frequencies, the condition $k_f a \gg \Omega_0$ is not achieved. This raises concern for the accuracy of the above limiting result when applied in the midfrequency range. In summary, Eqs. (10) and (4) are equivalent at frequencies that are large relative to the null frequency. Their relationship at frequencies comparable to the null frequency will be clarified in the next section.

II. NUMERICAL RESULTS

We consider steel shells in water, with $a/h=90$, $c_p=5435$, $\rho=7800$, $\nu=0.289$, $c_f=1482$, and $\rho_f=1000$, all in mks units. The ring, null, and coincidence frequencies for this set of parameters occur at $k_f a$ values of 3.67, 11.6, and 85.0, respectively. The coupling coefficients $G^{c,s}$ as computed from the three versions in Eqs. (4), (10), and (11) are summarized in Figs. 1 and 2 for the cylindrical and spherical shells, respectively. We note that the magnitude of the coupling coefficient is accurately predicted by both approximate methods. However, it is clear from Figs. 1 and 2 that the

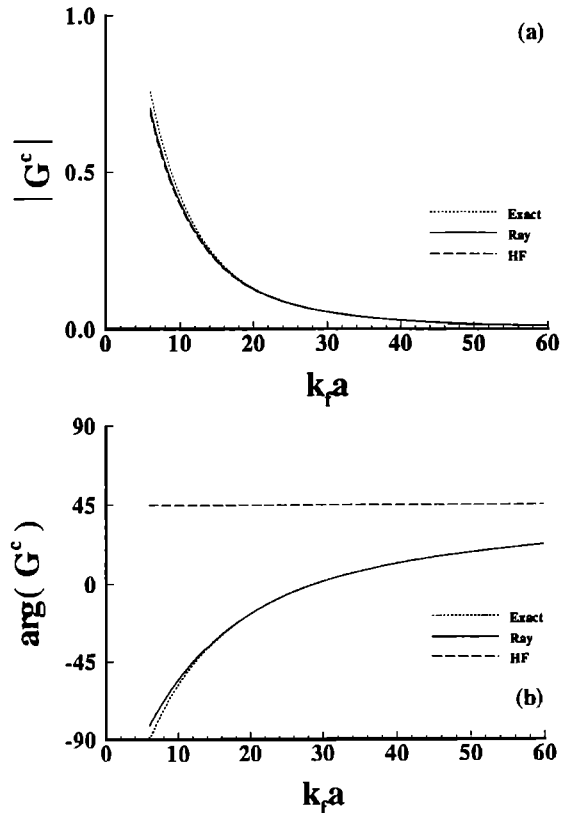


FIG. 1. (a) The magnitude and (b) phase of the coupling coefficient G^c for backscatter from an infinite circular cylindrical steel shell submerged in water, where the short-dashed line is Eq. (11), the solid line is Eq. (4), and the long-dashed line is Eq. (10).

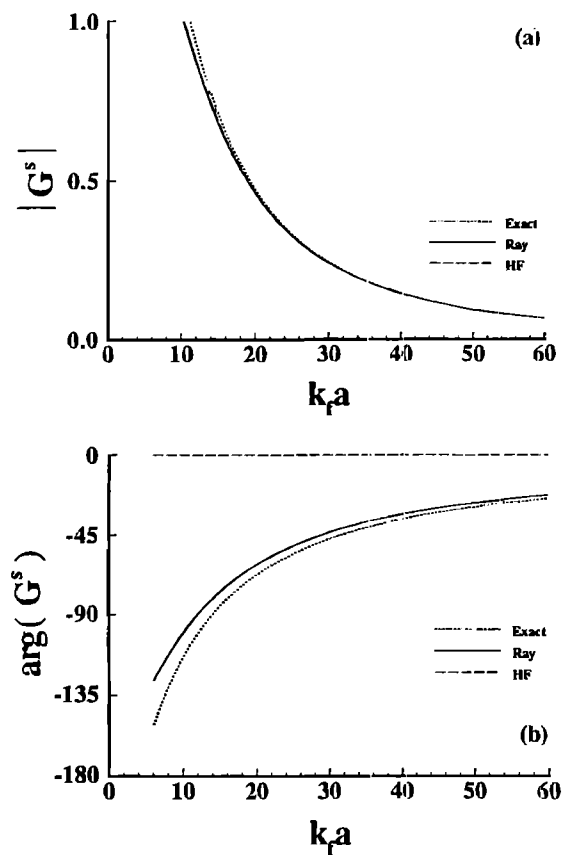


FIG. 2. As in Fig. 1, but for G^s , the spherical shell coupling coefficient.

phase of $G^{c,s}$ predicted by the high-frequency approximation of Eq. (10) disagrees with the exact solution for values of $k_f a$ between the ring and coincidence frequencies. At very high values of $k_f a$, the argument of $G^{c,s}$ given by Eq. (4) asymptotically approaches 45° for a cylindrical shell and 0° for a spherical shell, in agreement with the high-frequency limits in Eq. (10), and also with previous high-frequency studies.⁵

In summary, the ray-based representation of Eq. (4) is in very good agreement with the exact solution of Eq. (11) in both predicting the magnitude and the phase. This agreement for canonical shapes provides confidence in applying the general coupling coefficients for arbitrary shell curvature derived in Norris and Rebinsky² to determine scattering from more complicated structures.¹¹

III. CONCLUSIONS

The ray-theoretic results, which are based upon simple fundamental physical parameters, accurately predict both

magnitude and phase of the coupling coefficient defined with respect to the form function when compared to that based upon the Sommerfeld–Watson transformation. This provides a benchmark of the acoustic coupling theory for elastic shells of arbitrary shape developed by the authors.

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