

Acoustic and membrane wave interaction at plate junctions

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The interaction of an acoustic wave with two joined flat plates results in a diffracted acoustic wave emanating from the junction. The plates are assumed, for simplicity, to have different masses but no stiffness, and the exact solution of the two-dimensional diffraction problem is derived as a Fourier integral. The diffraction coefficient for a plane acoustic wave is found to depend in a simple manner upon the product of the total acoustic pressures at the junction for waves incident from the source and receiver directions. Also, the diffraction coefficient displays a maximum as a function of frequency, which is interpreted as a quasinresonance phenomenon. The results for the flat, massive plates are extended by perturbation methods to consider the acoustic interaction with longitudinal (membrane) waves originating from the junction of two curved shells, joined so that their tangent is continuous. Relatively simple results are found for the acoustic-to-membrane coupling coefficients, and these again show strong dependence on frequency. The related problem of membrane-to-acoustic diffraction is analyzed, and the diffraction coefficients obey simple reciprocity relations with the acoustic-to-membrane coefficients.

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INTRODUCTION

Any quantitative description of acoustic scattering from complex shaped elastic shells must invariably deal with various discontinuities which may be inherent to the structure. For example, there may be discontinuities due to abrupt changes in thickness, density, or radius of curvature. In this paper we describe a preliminary investigation of the acoustic scattering from a two-dimensional system of two joined curved plates. We are motivated by the results of a recent paper¹ in which it was demonstrated that the coupling and launching of structural membrane waves on smooth elastic shells of arbitrary shape can be separated from the scattering of the "background" acoustic response. The latter is relatively simple, as it involves only the inertial reaction of the structure to the acoustic pressure. The membrane waves are then driven, or forced, by the background field. Ray methods may then be used to obtain explicit expressions for the acoustic-to-membrane coupling and detachment coefficients for arbitrarily curved regions.¹ In this paper we first solve the exact diffraction problem for the "background" acoustic field scattered from two joined, flat compliant plates. This simplified model admits of a simple solution, which is then used to estimate the acoustic interaction with membrane waves at the junction of two curved plates. The central idea is that the wavelengths are short in comparison to the radii of curvature, leading to weak coupling between the acoustic and membrane wave fields. Thus the background solution for the flat plates provides both a good approximation to the acoustic response, but also it directly forces an ordinary differential equation which describes the scattered membrane wave field generated by the joint. Thereby, the acoustic wave and membrane wave interaction can be simply characterized. The flat plate model considered here explicitly ignores flexural effects, unlike the more precise studies of Brazier-Smith² and Norris and Wickham.³ We also approxi-

mate the coupling between acoustic and membrane waves for the curved plate model, using ray-theoretic arguments. Inclusion of flexural effects and the precise coupling between all the wave types leads to considerable analytical complexity, and is the topic of a separate paper.⁴

The diffraction of an acoustic wave from two joined massive plates is described in Sec. I. The scattered pressure generated by the joint is determined by solving a Wiener-Hopf problem. Similar Wiener-Hopf problems have been encountered in situations where the boundary conditions are of the impedance type considered here. Thus Crighton and Leppington⁵ analyzed the diffraction of an acoustic wave from a semi-infinite compliant plate with the same boundary conditions used here; Senior⁶ studied the diffraction by a semi-infinite metallic sheet of finite conductivity; Heins and Feshbach⁷ and Kay⁸ considered electromagnetic wave scattering by two half-planes; while Dahl and Frisk⁹ examined acoustic diffraction from the junction of a pressure release surface with a locally reacting half-plane. The solution to all of these problems depends upon the analytic factorization of the same type of Wiener-Hopf kernel. Here we tackle the factorization using an approach that differs from these authors, and yields relatively simple forms for the physically interesting quantities discussed in Sec. II, i.e., the junction pressure and the acoustic-acoustic diffraction coefficient. The flat plate results are extended in Sec. III to consider the acoustic interaction with membrane waves originating from the junction of two curved shells. The scattered membrane wave field is derived using perturbation methods and is given in terms of the tension in the curved plates. The diffracted tensions are determined by solving another Wiener-Hopf problem with a kernel related to that for the background pressure. Finally, in Sec. IV we consider the diffraction of an incoming membrane wave. Reciprocity is used to show that

the membrane-to-acoustic diffraction coefficients are equivalent to acoustic-to-membrane coefficients.

I. DIFFRACTION FROM TWO MASSIVE PLATES

A. Statement of the problem

We consider two plates extended along the x axis, with plate 1 on $x < 0$ and plate 2 on $x > 0$. The plates have no stiffness but their mass resists the fluid pressure through the time-harmonic force balance relating acoustic pressure p to the plate acceleration ($e^{-i\omega t}$ understood but omitted)

$$m_1 \omega^2 w = p, \quad x < 0, \quad y = 0, \quad (1a)$$

$$m_2 \omega^2 w = p, \quad x > 0, \quad y = 0, \quad (1b)$$

where $m_{1,2} = \rho_{1,2} h_{1,2}$ is the areal density of each plate. The displacement w is positive into the fluid. These simplified plate equations ignore bending, which is certainly significant at frequencies approaching the coincidence frequency. However, at lower frequencies we argue that the bending is unimportant. Brazier-Smith² and Norris and Wickham³ analyzed the problem with bending terms included, and their numerical results indicate that the flexural-to-acoustic coupling vanishes at lower frequencies. Therefore, on the basis of reciprocity, it is reasonable to expect that the acoustic-to-flexural interaction is also small at frequencies far below coincidence.

Upon substitution of Eqs. (1) into the continuity equation,

$$\rho_f \omega^2 w = \frac{\partial p}{\partial y}, \quad y = 0, \quad (2)$$

where ρ_f is the fluid density, one obtains impedance boundary conditions for the acoustic pressure,

$$\frac{1}{k_f} \frac{\partial p}{\partial y} = a_1 p, \quad x < 0, \quad y = 0, \quad (3a)$$

$$\frac{1}{k_f} \frac{\partial p}{\partial y} = a_2 p, \quad x > 0, \quad y = 0. \quad (3b)$$

Here a_1 and a_2 are dimensionless impedance parameters,

$$a_{1,2} = \rho_f / k_f m_{1,2}, \quad (4)$$

where $k_f = \omega / c_f$ is the fluid wave number, and c_f is the acoustic sound speed. Note that this definition of $a_{1,2}$ differs from that used by Norris and Wickham.^{3,4} Both a_1 and a_2 are large at low frequencies where the plates act effectively as a pressure release surface; whereas at high frequencies the impedance conditions are equivalent to rigid boundary conditions. The frequency at which $a = 1$ for a plate demarcates the transition between the low- and high-frequency regimes, and is known as the "null frequency." The present problem is therefore characterized by two distinct null frequencies, ω_1 and ω_2 , where

$$\omega_{1,2} = \rho_f c_f / m_{1,2}. \quad (5)$$

The problem statement is completed by noting that the pressure satisfies the Helmholtz equation

$$\nabla^2 p + k_f^2 p = 0, \quad y > 0. \quad (6)$$

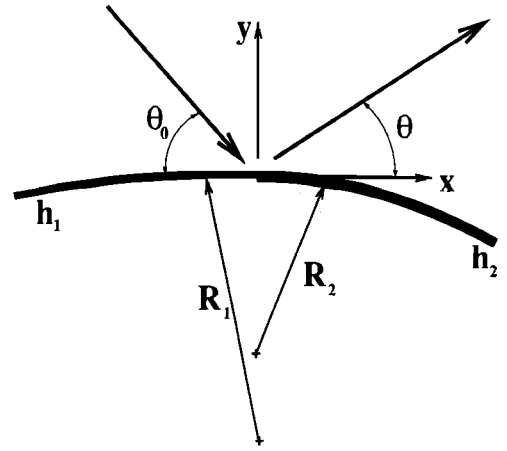


FIG. 1. The incident and scattered directions.

The incident wave field is chosen to be a plane wave with angle of incidence θ_0 , see Fig. 1,

$$p^i = e^{ik_f(x \cos \theta_0 - y \sin \theta_0)} = e^{ik_f r \cos(\theta + \theta_0)}, \quad (7)$$

where $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}(y/x)$ are polar coordinates. The specular wave field can be removed for $x < 0$ by first considering an incident plane wave of the form given by Eq. (7) which strikes a uniform plate composed entirely of a_1 . Then, the reflected wave field, p^r , which satisfies

$$\frac{1}{k_f} \frac{\partial}{\partial y} (p^i + p^r) = a_1 (p^i + p^r), \quad \text{on } y = 0, \quad (8)$$

can be readily determined as

$$p^r = \mathcal{R}_{1,2}(\theta_0) e^{ik_f r \cos(\theta - \theta_0)}, \quad (9)$$

where the acoustic reflection coefficients are

$$\mathcal{R}_{1,2}(\theta_0) = \frac{\sin \theta_0 - ia_{1,2}}{\sin \theta_0 + ia_{1,2}}. \quad (10)$$

Note that $\mathcal{R}_{1,2}(\theta_0)$ are of unit magnitude with phase increasing as a function of frequency, from $-\pi$ for $\omega \ll \omega_{1,2}$, to 0 for $\omega \gg \omega_{1,2}$.

The total pressure p is expressed as

$$p = p^i + p^r + p^s, \quad (11)$$

where p^s is the scattered wave field caused by the discontinuity at $x = 0$. This can be written as a spectrum of plane waves

$$p^s = \frac{1}{2\pi} \int_C S(\cos \beta) e^{ik_f r \cos(\theta - \beta)} d\beta, \quad (12)$$

where C is the contour in the β plane starting at $i\infty$, going to zero, then along the real axis across to π then out to $\pi - i\infty$. Alternatively, the scattered pressure can be written as a Fourier transform,

$$p^s = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{S(\xi)}{\gamma(\xi)} e^{k_f [i\xi x - \gamma(\xi)y]} d\xi, \quad y \geq 0, \quad (13)$$

where $\xi = \cos \beta$ and $\gamma(\xi) = -i\sqrt{1 - \xi^2}$, $|\xi| < 1$; or $\gamma(\xi) = \sqrt{\xi^2 - 1}$, $|\xi| > 1$.

B. Evaluation of the spectrum $S(\xi)$

Substitute the total pressure p given by Eqs. (7), (9), (11), and (12) into the boundary conditions given by Eqs. (3). One then obtains dual integral equations of the form

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \left(1 + \frac{a_1}{\gamma}\right) S(\xi) e^{ik_f \xi x} d\xi = 0, \quad x < 0, \quad (14a)$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \left(1 + \frac{a_2}{\gamma}\right) S(\xi) e^{ik_f \xi x} d\xi = \frac{2i\gamma_0(a_1 - a_2)}{\gamma_0 + a_1} e^{ik_f \xi_0 x}, \quad x > 0, \quad (14b)$$

where $\xi_0 = \cos \theta_0$ and the previous definition for γ implies $\gamma_0 = -i \sin \theta_0$. These dual integral equations will be solved using the standard Wiener–Hopf technique.¹⁰ Using Jordan's lemma the path of integration in Eq. (14a) may be closed in the lower complex plane without further contributing to the integral. Thus the integrand of the first integral equation should be analytic in the lower half-plane,

$$(1 + a_1/\gamma)S(\xi) = L(\xi), \quad (15)$$

where L is a function analytic in the lower half-plane. Following a similar argument for the second integral equation, (14b), the path of integration can be closed in the upper half-plane. Then, assuming the path of integration is indented below the pole at $\xi = \xi_0$, we have

$$\left(1 + \frac{a_2}{\gamma}\right)S(\xi) = \frac{2\gamma_0(a_1 - a_2)}{\gamma_0 + a_1} \frac{U(\xi)}{U(\xi_0)} \frac{1}{\xi - \xi_0}, \quad (16)$$

where U is a function analytic in the upper complex half-plane.

In the process of determining S from Eqs. (15) and (16), the expression

$$K(\xi) = \frac{\gamma(\xi) + a_1}{\gamma(\xi) + a_2}, \quad (17)$$

must be represented as the product of functions analytic in the lower and upper complex ξ plane. Thus

$$K(\xi) = K_-(\xi)/K_+(\xi), \quad (18)$$

where K_+ is analytic in the upper ξ plane and K_- is analytic in the lower. Note that K is an even function, implying the symmetry

$$K_+(\xi) = 1/K_-(-\xi). \quad (19)$$

Similar functions have been considered by Senior,⁶ Heins and Feshbach,⁷ Kay,⁸ and Dahl and Frisk.⁹ In each of these papers the authors were concerned with values of impedance parameters $a = a_1$ or a_2 that were not real and positive. In calculating the split functions, one method is to consider the logarithm of the kernel. To recover the kernel, it is necessary to integrate this result along a suitable path ending at ξ , which is real-valued for cases of interest. In each of the previous treatments,^{6–8} the parameter a was complex so that the integration could be done along the real axis. When a is real, poles lie on the real axis so that a different path of integration must be chosen increasing the complexity of the method. This was not an issue for Dahl and Frisk⁹ who approximated the split functions. By using a transformation of

the parameter appearing in the integral representation of the kernel, a simple method of analytic splitting is obtained and is shown in Appendix A for real and positive impedance parameters.

In order to determine L and subsequently S , Eqs. (15) and (16) are rewritten so that each side of the equation is analytic in only one half-plane. Doing so,

$$\frac{K_-(\xi_0)}{K_-(\xi)} L(\xi)(\xi - \xi_0) = \frac{2\gamma_0(a_1 - a_2)}{\gamma_0 + a_2} \times \frac{1}{K_+(\xi)K_+(\xi_0)} \frac{U(\xi)}{U(\xi_0)}, \quad (20)$$

and it may be shown that each side is constant. The constant is evaluated by substituting $\xi = \xi_0$ into the right member, yielding an expression for $L(\xi)$. The spectral function S describing the scattered wave field is then obtained using Eq. (15) as

$$S(\xi) = \frac{2(a_1 - a_2)K_+(\xi)}{(1 + a_2/\gamma_0)(1 + a_2/\gamma)K_-(\xi_0)(\xi - \xi_0)}. \quad (21)$$

The symmetric dependence upon ξ and $-\xi_0$ [cf. Eq. (19)] implies that the solution satisfies acoustical reciprocity under the interchange of source and receiver directions. In summary, the total pressure, Eq. (11), is given by

$$p = e^{ik_f r \cos(\theta + \theta_0)} + \mathcal{H}_1(\theta_0) e^{ik_f r \cos(\theta - \theta_0)} + p^s(r, \theta), \quad (22)$$

where

$$p^s = \frac{(a_1 - a_2)}{K_-(\cos \theta_0)} \frac{\sin \theta_0}{\sin \theta_0 + ia_2} \frac{1}{\pi} \times \int_C \frac{\sin \beta}{\sin \beta + ia_2} \frac{K_+(\cos \beta)}{\cos \beta - \cos \theta_0} e^{ik_f r \cos(\theta - \beta)} d\beta. \quad (23)$$

This completes the formal derivation of the diffraction problem. We now discuss some of its properties in the near and far fields.

II. DISCUSSION OF THE GENERAL SOLUTION

A. The junction pressure

The general expression (23) allows us to determine the pressure at and near the junction $x = 0$. The behavior for small x depends upon the asymptotic behavior of the transform P for $|\xi| \rightarrow \infty$, where the full and half-transforms for the pressure are defined by

$$P(\xi) = P_+(\xi) + P_-(\xi) = \frac{S(\xi)}{i\gamma(\xi)}. \quad (24)$$

Here S is the spectral decomposition for the scattered pressure p^s given by Eq. (21), and P_+ and P_- may be viewed as a sum decomposition of the function $S/i\gamma$, analytic in the upper and lower half-planes, respectively. They may be defined as

$$P_+(\xi) = \int_{-\infty}^0 p^s(x, 0) e^{-ik_f \xi x} k_f dx, \quad (25a)$$

$$P_-(\xi) = \int_0^\infty p^s(x, 0) e^{-ik_f \xi x} k_f dx. \quad (25b)$$

Since the pressure is continuous across the joint, its value there may be determined using P_+ evaluated for large wave number through the relation

$$P_+(\xi) = \frac{i}{\xi} p^s(0) + O(\xi^{-2}), \quad |\xi| \rightarrow \infty. \quad (26)$$

Thus

$$p^s(0) = \lim_{\xi \rightarrow \infty} -i\xi P_+(\xi). \quad (27)$$

The partition of P can be achieved by standard means. We first note that the presence of γ in Eq. (24) can be removed by using the identity $(\gamma + a_2)/K_+ = (\gamma + a_1)/K_-$ to write $P(\xi)$ in two different ways, and then eliminate γ between them. Doing so yields

$$P(\xi) = \frac{2i}{(1 + a_2/\gamma_0)K_-(\xi_0)} [K_+(\xi) - K_-(\xi)] \frac{1}{\xi - \xi_0}. \quad (28)$$

The sum decomposition of P is obtained by transferring the pole at $\xi = \xi_0$, giving

$$P_\pm(\xi) = \frac{\pm 2i\gamma_0}{\gamma_0 + a_2} \left(\frac{K_\pm(\xi)}{K_\pm(\xi_0)} - \frac{\gamma_0 + a_2}{\gamma_0 + a_1} \right) \frac{1}{\xi - \xi_0}, \quad (29)$$

Note that $P_+ \rightarrow -P_-$ as $|\xi| \rightarrow \infty$. It follows from (26) that

$$p^s(0) = \frac{2\gamma_0}{(\gamma_0 + a_2)K_-(\xi_0)} - [1 + \mathcal{R}_1(\theta_0)]. \quad (30)$$

The total pressure at the joint can now be written as

$$p(0) = p_0(\theta_0), \quad (31)$$

where

$$p_0(\theta_0) = \frac{1 + \mathcal{R}_1(\theta_0)}{K_+(\xi_0)} = \frac{1 + \mathcal{R}_2(\theta_0)}{K_-(\xi_0)}. \quad (32)$$

A more explicit form follows by using Eqs. (19) and (A10),

$$p_0(\theta_0) = \frac{2 \sin \theta_0}{\sqrt{(\alpha_1 - \cos \theta_0)(\alpha_2 + \cos \theta_0)}} \times [\mathcal{R}_1(\theta_0) \mathcal{R}_2(\theta_0)]^{1/4} e^{i[I_2(\cos \theta_0) - I_1(\cos \theta_0)]}, \quad (33)$$

where $\alpha_{1,2} = \sqrt{1 + a_{1,2}^2}$, and the quantity $I(\xi)$ is defined in (A11). Note that the latter is real-valued for plane-wave incidence, and therefore, since the reflection coefficients are of unit amplitude, the magnitude of the total pressure at the junction is simply

$$|p_0(\theta_0)| = \frac{2 \sin \theta_0}{\sqrt{(\alpha_1 - \cos \theta_0)(\alpha_2 + \cos \theta_0)}}. \quad (34)$$

Hence, $0 \leq |p_0(\theta_0)| \leq 2$. When the plates are identical, i.e., $a_1 = a_2$, then $K_+, K_- \rightarrow 1$ and the pressure at the joint reduces to $p(0) = 1 + \mathcal{R}_1(\theta_0) = 1 + \mathcal{R}_2(\theta_0)$, as one might expect in this simple limit.

B. The reflected and diffracted pressure

The total response in the fluid, defined by Eqs. (22) and (23), simplifies in the far field. First, the pole at $\beta = \theta_0$ contributes a specularly reflected wave field, p_s^s , which simplifies to

$$p_s^s = [\mathcal{R}_2(\theta_0) - \mathcal{R}_1(\theta_0)] e^{ik_f r \cos(\theta - \theta_0)} H(\theta_0 - \theta), \quad (35)$$

where H is the Heaviside step function, and $\mathcal{R}_2$ is defined in Eq. (10). Hence, we obtain the correct reflected response from the plate $x > 0$ in the appropriate region of the fluid.

In addition to these specular effects a diffracted wave field, p_d^s , results from the stationary phase point of the integral at $\beta = \theta$. A diffraction coefficient $D(\theta, \theta_0)$ can be defined for the far field by

$$p_d^s = D(\theta, \theta_0) \sqrt{2/\pi k_f r} e^{-i\pi/4} e^{ik_f r}, \quad k_f r \rightarrow \infty, \quad 0 < \theta \leq \pi. \quad (36)$$

This diffraction coefficient can be found from Eq. (23) by first deforming the contour of integration to the path of steepest descent, $\hat{C}(\theta)$, and then using the asymptotic integral approximation¹¹

$$\int_{\hat{C}(\theta)} G(\cos \beta) e^{ik_f r \cos(\theta - \beta)} d\beta \approx G(\cos \theta) \sqrt{\frac{2\pi}{k_f r}} e^{-i\pi/4} e^{ik_f r}, \quad k_f r \rightarrow \infty. \quad (37)$$

The diffraction coefficient is therefore

$$D(\theta, \theta_0) = \frac{(a_1 - a_2)}{\cos \theta - \cos \theta_0} \frac{\sin \theta_0}{\sin \theta_0 + ia_2} \times \frac{K_+(\cos \theta)}{K_-(\cos \theta_0)} \frac{\sin \theta}{\sin \theta + ia_2}, \quad (38)$$

which may be simplified, using the definition of p_0 in Eq. (33), to the concise form

$$D(\theta, \theta_0) = \frac{1}{4} \left(\frac{a_1 - a_2}{\cos \theta - \cos \theta_0} \right) p_0(\pi - \theta) p_0(\theta_0). \quad (39)$$

The diffraction coefficient clearly satisfies the symmetries

$$D(\theta, \theta_0) = D(\pi - \theta_0, \pi - \theta), \quad (40)$$

$$D(a_1, a_2; \theta, \theta_0) = D(a_2, a_1; \pi - \theta, \pi - \theta_0), \quad (41)$$

the first of which is required by acoustical reciprocity, and the second is a statement of the fact that the problem is identical when viewed in a mirror. Also, $D(\theta, \theta_0)$ is singular at the reflection boundary given by $\theta = \theta_0$. This is natural, and indicates the transition in the specular field from one reflection coefficient to another. A uniformly asymptotic description can be obtained by standard means. However, we focus on the diffraction coefficient because this dominates the response in directions away from the specular.

It is useful to examine the behavior of the diffraction coefficient as a function of frequency. For example, the magnitude and phase of D is shown in Fig. 2. We note in particular the appearance of a maximum for $|D|$, which is a characteristic feature. The impedance parameters depend

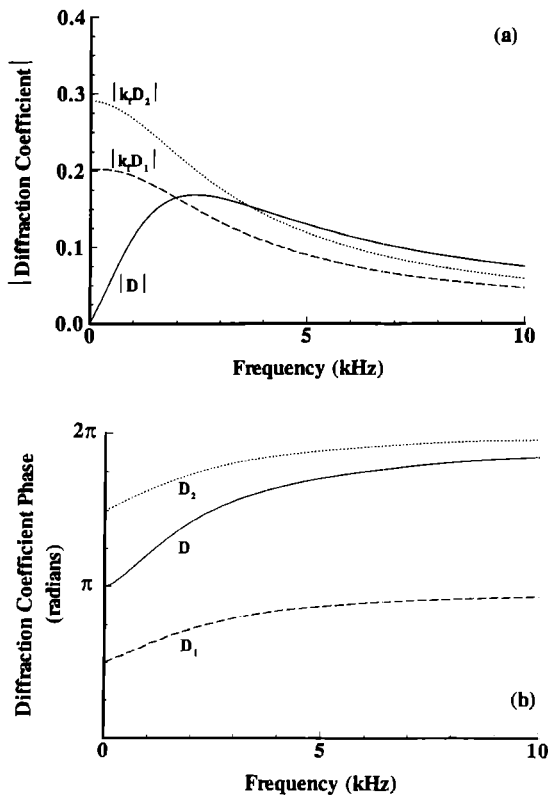


FIG. 2. The magnitude, (a), and phase, (b), of the diffraction coefficients of Eqs. (39) and (70). The plates are both steel in water with $c_f=1482$, $c_p=5435$, $\rho_f=1000$, and $\rho=7800$, all in mks units, where c_p is the longitudinal wave speed in steel [$c_p=\omega/k_{1,2}$, see Eq. (71)]. The plate thicknesses are $h_1=0.0254$ m and $h_2=0.0127$ m. The coefficient D is given for angles of incidence and observation of $\theta_0=30^\circ$ and $\theta=130^\circ$. The acoustic-to-membrane coefficients are for the same incidence, and for plate radii of curvature $R_1=2$ m and $R_2=8$ m. The coefficients D_1 and D_2 have dimensions of length, so we plot the dimensionless quantities $|k_j D_j|$, $j=1,2$.

upon the frequency, because Eqs. (4) and (5) imply $a_{1,2}=\omega_{1,2}/\omega$. They therefore become large in the low-frequency limit, which may be defined as $\omega \ll \min(\omega_1, \omega_2)$. The diffraction coefficient simplifies in this limit to

$$D(\theta, \theta_0) \approx \frac{\sin \theta \sin \theta_0}{\cos \theta - \cos \theta_0} \left(\frac{1}{a_1} - \frac{1}{a_2} \right), \quad (42)$$

which is small in magnitude, of order ω . The high-frequency regime is defined in the same way by $\omega \gg \max(\omega_1, \omega_2)$, and the asymptotic form of the diffraction coefficient is

$$D(\theta, \theta_0) \approx \frac{a_1 - a_2}{\cos \theta - \cos \theta_0}, \quad (43)$$

which is also small, this time of order ω^{-1} . Note that the limiting low- and high-frequency approximations, (42) and (43), are both real but of opposite sign. They are also small, which begs the question of whether they can ever be of order unity. The answer is in the affirmative, as can be seen by considering the case of $a_2 \gg 1 \gg a_1$, for which

$$D(\theta, \theta_0) \approx i2 \frac{\cot(\theta/2) \tan(\theta_0/2)}{\cos \theta - \cos \theta_0}. \quad (44)$$

Note that the diffraction coefficient is purely imaginary in (44). We could call this the “resonance regime,” because

both the amplitude is relatively large compared with (42) and (43), and the phase is 90° out of phase with both. In general, the limiting forms in (42) and (43) imply that there must be some frequency at which D is purely imaginary, which we define as the *impedance resonance frequency* (or frequencies). The limiting low- and high-frequency forms for D can be found by simpler means, taking advantage of the asymptotic behavior of the impedance parameters $a_{1,2}$ in these limits. The procedure is outlined in Appendix B.

III. ACOUSTIC COUPLING TO MEMBRANE WAVES

A. Relation to the diffraction problem

Consider two curved plates joined with continuous tangent along the z axis with different densities, thicknesses, and curvatures which are fluid loaded on one side only, the side with $y > 0$. Their behavior may be described individually by the equations of motion for a two-dimensional shell. Let v and w be the in-surface and normal (into the fluid) displacements, and p the total acoustic pressure in the fluid, then the time-harmonic boundary conditions are

$$\tau_{,s} + m\omega^2 v = 0, \quad (45a)$$

$$\tau/R + Bw_{,sss} - m\omega^2 w = -p, \quad (45b)$$

where s is the arc length, R is the radius of curvature, B is the bending stiffness, τ is the tensile or longitudinal stress in the plate,

$$\tau = C(v_{,s} + w/R), \quad (46)$$

and C is the extensional stiffness. Also, m is the mass per unit area, as before, and the stiffnesses C and B can be related to the intrinsic properties of the plate; thus $C = Eh/(1 - \nu^2)$ and $B = Eh^3/12(1 - \nu^2)$, where h , E , and ν are the thickness, Young's modulus, and Poisson ratio, respectively. Using Eq. (46), the first equation, (45a), can be rewritten as

$$\tau_{,ss} + k^2 \tau = (m\omega^2/R)w, \quad (47)$$

where $k = \omega\sqrt{m/C}$ is the longitudinal, or membrane, wave number. The equation of kinematic continuity between the plate and the fluid is given by Eq. (2) with the y derivative replaced by the normal derivative at each point.

We follow the method outlined in Norris and Rebinsky¹ which scaled displacements in a Fresnel or “coupling” zone in a study of acoustic coupling of membrane waves to a smooth elastic shell. Here we scale the in plane and out-of-plane displacements in a local zone about the joint of the two curved plates. In this region, each curved plate is assumed to have its shape described by $y + x^2/2R \approx 0$ with its arc length given by $s = x + x^3/6R^2 + \dots$, where R is a typical radius of curvature. It is assumed that R is much larger than the fluid wavelength. Define the parameter $\epsilon = \max \epsilon_{1,2}$, where

$$\epsilon_{1,2} = 1/k_f R_{1,2}, \quad (48)$$

and it is explicitly assumed that $\epsilon \ll 1$. Next a slow scale may be defined through the dimensionless variables $\mathbf{X} = k_f \mathbf{x}$ and $S = k_f s$. Then $X = S[1 + O(\epsilon^2)]$ so that the “inner region” is defined over one wavelength or $S = O(1)$. Also, the incident wave field of Eq. (7) can be scaled on the surface as

$$p^j \approx e^{i(X \cos \theta_0 + 1/2 \epsilon X^2 \sin \theta_0)} \approx e^{iX \cos \theta_0} + O(\epsilon), \quad (49)$$

using the same approximations. Therefore, curvature can be ignored to leading order so that each curved plate may be “flattened” about the junction.

Assume that the frequency of excitation is below the coincidence frequency of each curved plate, then flexural terms can also be neglected. If the angle of incidence is not near the critical angle of longitudinal waves, then the v displacement is small and the following *ansatz*¹ is assumed,

$$(p, w) = (p^{(0)}, w^{(0)}) + \epsilon(p^{(1)}, w^{(1)}) + \dots, \quad (50a)$$

$$(v, \tau) = \epsilon^\alpha(v^{(0)}, \tau^{(0)}) + \epsilon^{2\alpha}(v^{(1)}, \tau^{(1)}) + \dots, \quad (50b)$$

where $\alpha > 0$ is a parameter used to scale the equations. In practice, we set $\alpha = 0$, so that $\tau^{(0)}$, for example, is the leading order membrane stress. The $p^{(0)}$ and $w^{(0)}$ terms define the *background solution*. Note that if the plates are the same the leading order diffraction is determined by $p^{(1)}$.

The leading-order boundary conditions are found by using Eq. (50) in Eq. (45), and are given by Eq. (1) but now for $p^{(0)}$ rather than p . Therefore, the leading-order pressure field satisfies the impedance conditions (3), and we can therefore think of this problem as solved, with the pressure as known, being given by Eqs. (22) and (23). Using this solution for the background pressure $p^{(0)}$, the τ equation given by Eq. (47) can be rewritten as

$$\tau_{,xx}^{(0)} + k^2 \tau^{(0)} = (1/R)p^{(0)}, \quad y = 0, \quad (51)$$

for each “curved” plate. Thus the dynamic tensions in the plates, which include the longitudinal membrane waves diffracted from the joint at $x = 0$, are forced by the pressure $p^{(0)}$. The forcing follows from Eqs. (10) and (11) as

$$\frac{1}{R} p^{(0)} = \frac{1}{R} [1 + \mathcal{R}_1(\theta_0)] e^{-ik_f \xi_0 x} + \frac{1}{R} p^s(x). \quad (52)$$

The major difficulty arises from the final term, which follows from (23). However, it vanishes identically if $a_1 = a_2$, or equivalently, if $m_1 = m_2$. We will see how the solution for this simple case drops out of the general solution.

B. The general solution

To determine $\tau^{(0)}$, we follow a procedure similar to that used for the acoustic diffraction problem. The specular stress field on a homogeneous plate of either material 1 or 2 is of the form $T_j \exp(ik_f \xi_0 x)$, $j = 1$ or 2 , where the amplitude follows from Eqs. (11) and (51) as

$$k_f T_j = [\epsilon_j / F_j(\xi_0)] [1 + \mathcal{R}_j(\theta_0)], \quad j = 1 \text{ or } 2, \quad (53)$$

where $\epsilon_{1,2}$ are given in Eq. (48). The dispersion relations $F_{1,2}(\xi)$ are defined as

$$F_{1,2}(\xi) = \xi_{1,2}^2 - \xi^2, \quad (54)$$

where the dimensionless quantities $\xi_{1,2}$ are defined by

$$k_{1,2} \equiv k_f \xi_{1,2}. \quad (55)$$

We therefore assume a solution in the form of the specular wave field on plate 1 plus an additional scattered field,

$$\tau^{(0)} = T_1 e^{ik_f \xi_0 x} + \tau^s(x), \quad -\infty < x < \infty. \quad (56)$$

Thus Eq. (51) becomes

$$\tau_{,xx}^s + k_1^2 \tau^s - (1/R_1)p^s = 0, \quad x < 0, \quad (57a)$$

$$\tau_{,xx}^s + k_2^2 \tau^s - (1/R_2)p^s = k_f \Gamma e^{ik_f \xi_0 x}, \quad x > 0, \quad (57b)$$

where Γ is defined as

$$\Gamma = \left(\epsilon_2 - \epsilon_1 \frac{F_2(\xi_0)}{F_1(\xi_0)} \right) [1 + \mathcal{R}_1(\theta_0)]. \quad (58)$$

The scattered pressure p^s is again assumed to have the representation in Eq. (13), and the scattered tensile force τ^s is assumed to be

$$\tau^s(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} T(\xi) e^{ik_f \xi x} d\xi. \quad (59)$$

The transform $T(\xi)$ can be partitioned in the same manner as before for the pressure transform, i.e., $T = T_+ + T_-$ where $T_{\pm}$ are each regular in a half-plane of the ξ -complex plane. The half-transforms of derivatives are obtained through the use of integration by parts. Applying the generalized transforms defined in the above to Eqs. (57) gives

$$F_1(\xi) k_f T_+(\xi) = \epsilon_1 P_+(\xi) - \tau_{,x}^s(0_-) - ik_f \xi \tau^s(0_-), \quad (60a)$$

$$F_2(\xi) k_f T_-(\xi) = \epsilon_2 P_-(\xi) + \tau_{,x}^s(0_+) + ik_f \xi \tau^s(0_+) - i\Gamma / (\xi - \xi_0). \quad (60b)$$

Divide the first equation by $F_1(\xi)$, giving an expression for T_+ . However, the rhs then has an apparent pole at $\xi = \xi_1$ in the upper half-plane, but its residue must be zero. Similarly, dividing the second equation by $F_2(\xi)$ and setting the residue of the pole at $\xi = -\xi_2$ to zero, we obtain the dual identities

$$\tau_{,x}^s(0_-) + i\xi_1 k_f \tau^s(0_-) = \epsilon_1 P_+(\xi_1), \quad (61a)$$

$$\tau_{,x}^s(0_+) - i\xi_2 k_f \tau^s(0_+) = -\epsilon_2 P_-(\xi_2) - \frac{i\Gamma}{\xi_2 + \xi_0}. \quad (61b)$$

These can be combined with Eqs. (60) to eliminate $\tau_{,x}^s(0_{\pm})$, yielding for $T_{\pm}$:

$$k_f T_+(\xi) = \epsilon_1 \frac{P_+(\xi) - P_+(\xi_1)}{F_1(\xi)} + \frac{ik_f \tau^s(0_-)}{\xi + \xi_1}, \quad (62a)$$

$$k_f T_-(\xi) = \epsilon_2 \frac{P_-(\xi) - P_-(\xi_2)}{F_2(\xi)} - \frac{ik_f \tau^s(0_+)}{\xi - \xi_2} + \frac{i\Gamma}{F_2(\xi_0)} \left(\frac{1}{\xi - \xi_2} - \frac{1}{\xi - \xi_0} \right). \quad (62b)$$

We consider the edge conditions for a welded junction, implying continuity of both displacement and stress,

$$v^{(0)}(0_-) = v^{(0)}(0_+), \quad \tau^{(0)}(0_-) = \tau^{(0)}(0_+). \quad (63)$$

The latter implies, with Eqs. (62), that both T_+ and T_- depend upon only a single unknown, $\tau^s(0) = \tau^{(0)}(0) - T_1$. This can be found by noting that the expansions of $T_{\pm}$ at the point at infinity are related to the behavior of the tension at the joint, cf. Eq. (26). Thus

$$\pm T_{\pm}(\xi) = \frac{i}{\xi} \tau^s(0\mp) + \frac{1}{\xi^2 k_f} \tau_{,x}^s(0\mp) + o(\xi^{-2}). \quad (64)$$

The in-surface displacement follows from Eq. (45a) as

$$v^{(0)}(x) = -(m\omega^2)^{-1} \tau_{,x}^{(0)}(x), \quad (65)$$

and therefore the remaining continuity condition, (63)₁, can be expressed as

$$\lim_{|\xi| \rightarrow \infty} \{ \xi^2 [a_1 T_+(\xi) + a_2 T_-(\xi)] - i\xi(a_1 - a_2) \tau^s(0) \} = i\xi_0(a_2 - a_1) T_1, \quad (66)$$

where we have used Eqs. (56) and (65). Substituting for T_+ and T_- from Eqs. (62) yields an explicit expression for $\tau^s(0)$. After some simplification, and using (29) and (32), we find that the total tension at the junction is

$$k_f \tau^{(0)}(0) = \frac{p_0(\theta_0)}{a_1 \xi_1 + a_2 \xi_2} \left(\epsilon_1 a_1 \frac{K_+(\xi_1)}{\xi_1 - \xi_0} + \epsilon_2 a_2 \frac{K_-(-\xi_2)}{\xi_2 + \xi_0} \right). \quad (67)$$

It is then straightforward to show that the in-surface velocity at the junction is

$$-i\omega v^{(0)}(0) = -\frac{a_1 a_2}{\rho_f c_f} \frac{p_0(\theta_0)}{a_1 \xi_1 + a_2 \xi_2} \left(\epsilon_1 \xi_2 \frac{K_+(\xi_1)}{\xi_1 - \xi_0} - \epsilon_2 \xi_1 \frac{K_-(-\xi_2)}{\xi_2 + \xi_0} \right). \quad (68)$$

C. The scattered tension

The scattered tension due to the presence of the joint at $x=0$ is obtained from Eqs. (59) and (62). The scattered solution for $x>0$ and $x<0$, depends upon the functions T_- and T_+ , respectively, and the physically significant poles of these half-transforms are at $\xi=\xi_0$ and ξ_2 for $x>0$, and at $\xi=-\xi_1$ for $x<0$. By considering the residue contributions from these poles and using Eqs. (29), (53), and (56), we can reduce the total tension to the following form

$$\tau^{(0)}(x) = T_j e^{ik_f \xi_0 x} + D_j e^{ik_f \xi_j |x|} + \tau_{\text{near}}(x), \quad (69)$$

where $j=1,2$ corresponds to $x<0$ and $x>0$, respectively. The first term corresponds to the field on each plate in the absence of the other, and the coefficients D_1 and D_2 give the diffracted longitudinal wave amplitudes. The final term, τ_{near} , denotes the remainder which is a Fourier integral of a function with no physically significant poles. It therefore vanishes far from the joint, and is important only in the near field. Evaluating the residues and simplifying, yields

$$D_1(\theta_0) = \tau^{(0)}(0) - \frac{\epsilon_1 p_0(\theta_0)}{k_f 2 \xi_1} \left(\frac{K_+(\xi_1)}{\xi_1 - \xi_0} + \frac{K_+(-\xi_1)}{\xi_1 + \xi_0} \right), \quad (70a)$$

$$D_2(\theta_0) = \tau^{(0)}(0) - \frac{\epsilon_2 p_0(\theta_0)}{k_f 2 \xi_2} \left(\frac{K_-(-\xi_2)}{\xi_2 - \xi_0} + \frac{K_-(-\xi_2)}{\xi_2 + \xi_0} \right). \quad (70b)$$

The frequency dependence of the diffraction coefficients is illustrated in Figs. 2 and 3. We note that both D_1 and D_2

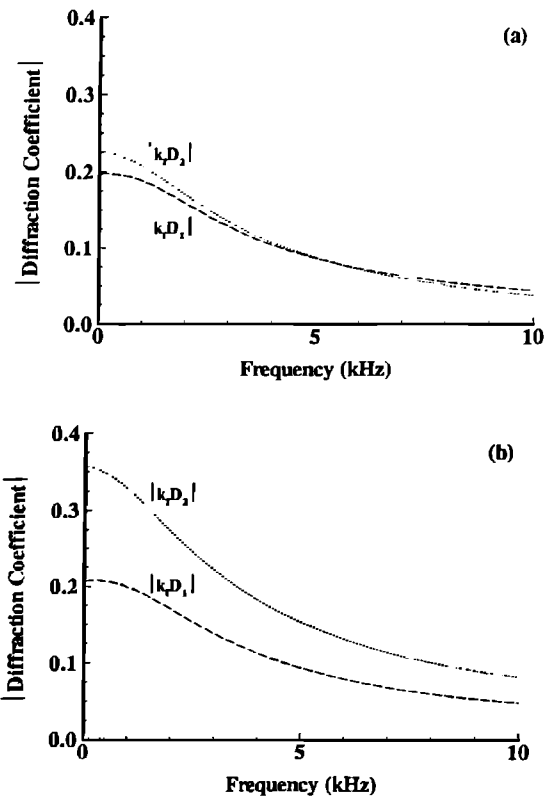


FIG. 3. The magnitude of the longitudinal wave diffraction coefficients D_1 and D_2 for the same parameters as in Fig. 2, except $R_2=4$ m in (a), and $R_2=\infty$ in (b).

vanish when both plates are flat, because there is no coupling to the in-surface motion in this limit. However, if one plate is flat and the other curved, then longitudinal waves are diffracted onto both plates, but the amplitude on the flat plate is relatively simple: thus *the diffraction coefficient for a flat plate is simply the total tension at the junction*, as given by Eq. (67).

Note that D_1 becomes unbounded for $\theta_0=\theta_1$, $\pi-\theta_1$, or $\pi-\theta_2$, where the membrane angles are defined by

$$\cos \theta_{1,2} = \xi_{1,2} = k_{1,2}/k_f. \quad (71)$$

These singularities in the solution are not unexpected because the model equations (57) predict exact resonances at these angles of incidence. Similarly, D_2 blows up for $\theta_0=\theta_2$, $\pi-\theta_2$, or θ_1 . The infinite resonances occur when the incident wave is exactly in phase with the free mode of the plate, and because the equations as stated apply for the full range of $-\infty < x < \infty$. This effect is an artifact of the approximate equations (57), which were derived on the assumption that the v displacement is small, $O(\epsilon)$, in comparison to the pressure p and normal displacement w , see Eq. (50). When the angle of incidence θ_0 is close to one of the membrane angles a plate wave of $O(\sqrt{\epsilon})$ is excited by the in-phase forcing of the acoustic wave on the membrane equations. The details of this mechanism and the meaning of “close” in the previous sentence are discussed by Norris and Rebinsky.¹ It is important to realize that the “resonant” plate wave, while large compared with the assumed scaling of Eq. (50), is still actually small, $O(\sqrt{\epsilon})$,¹ in distinct contrast to the infinite reso-

nance predicted by the “flattened” equations (57). Furthermore, the membrane wave on a curved shell is excited over a finite region defined by a boundary layer near the specular point, whereas the approximation used here assumes that the excitation is of infinite extent. The present analysis is therefore invalid when the singular terms give $\epsilon/(\cos \theta_0 \pm \cos \theta_{1,2}) = O(\sqrt{\epsilon})$, which occurs when $\cos \theta_0 \pm \cos \theta_{1,2} = O(\sqrt{\epsilon})$. Alternatively, D_1 and D_2 are accurate for $\cos \theta_0 \pm \cos \theta_{1,2} = O(1)$.

Finally, we return to the limiting case of $m_1 = m_2$ or equivalently, $a_1 = a_2$. Equation (69) still applies, but now the near-field τ_{near} is identically zero, and the diffracted amplitudes are simply

$$D_j(\theta_0) = \tau^{(0)}(0) - T_j, \quad j = 1, 2, \quad (72)$$

where the total tension at the junction is

$$\tau^{(0)}(0) = \left(\frac{\xi_1 + \xi_0}{\xi_1 + \xi_2} \right) T_1 + \left(\frac{\xi_2 - \xi_0}{\xi_1 + \xi_2} \right) T_2. \quad (73)$$

These results follow from the general solution by noting that both K_+ and K_- are unity in this limit. They can also be derived by solving Eq. (57) directly, noting that $p^s = 0$, and then applying the continuity conditions (63). We note that if the plates also have the same longitudinal wave speeds, then $\theta_1 = \theta_2$ in addition to $a_1 = a_2$, and the diffracted amplitudes simplify to

$$D_j(\theta_0) = (\epsilon_2 - \epsilon_1) \frac{1 + \mathcal{R}_1(\theta_0)}{2 \cos \theta_1} \times \begin{cases} (\cos \theta_0 + \cos \theta_1)^{-1}, & j = 1, \\ (\cos \theta_0 - \cos \theta_1)^{-1}, & j = 2. \end{cases} \quad (74)$$

The membrane diffraction vanishes when the curvatures are identical ($\epsilon_1 = \epsilon_2$).

IV. DIFFRACTION OF AN INCOMING MEMBRANE WAVE

A. General solution

The same type of asymptotic approximations used in the previous section can be applied to the related problem of membrane-to-acoustic diffraction at the junction of two curved shells. We consider an incident membrane wave with plate tension τ^j of the form $e^{ik_f \xi_1 x}$ incoming from plate 1, and striking the joint at $x = 0$. Both reflected and transmitted membrane waves along with a diffracted pressure wave field are produced. However, in keeping with the decoupling arguments of the previous section, we can first approximate the plate tension field as if the fluid were absent. The reflected wave then has the form $\Gamma_R e^{-ik_f \xi_1 x}$ and the transmitted wave is $\Gamma_T e^{ik_f \xi_2 x}$ where the coefficients follow by matching the tension and velocity at $x = 0$ [cf. Eq. (63)]

$$\Gamma_R = \frac{a_1 \xi_1 - a_2 \xi_2}{a_1 \xi_1 + a_2 \xi_2}, \quad \Gamma_T = \frac{2a_1 \xi_1}{a_1 \xi_1 + a_2 \xi_2}. \quad (75)$$

The diffracted acoustic pressure is then found by considering this leading-order tension field, denoted by $\tau^{(0)}$, as the driving mechanism for the leading-order pressure, $p^{(0)}$. The coupling is via the w equation of (45b), which reduces to (neglecting the bending term as before) the following forced impedance boundary conditions for the pressure

$$\frac{1}{k_f} \frac{\partial p^{(0)}}{\partial y} - a_1 p^{(0)} = \frac{a_1}{R_1} \tau^{(0)}, \quad x < 0, \quad (76a)$$

$$\frac{1}{k_f} \frac{\partial p^{(0)}}{\partial y} - a_2 p^{(0)} = \frac{a_2}{R_2} \tau^{(0)}, \quad x > 0. \quad (76b)$$

In summary, the tension $\tau^{(0)}$ on plate 1 is composed of incident and reflected membrane waves and on plate 2 there is a transmitted membrane wave. These boundary conditions, combined with the Helmholtz equation (6), determine the leading-order acoustic field.

Once again we choose a solution of the form of a “specular” wave field on plate 1 plus an additional scattered field,

$$p^{(0)} = P_1 e^{ik_f [\xi_1 x + \gamma(\xi_1)y]} + p^s, \quad -\infty < x < \infty, \quad (77)$$

where P_1 is chosen so that Eq. (76a) is satisfied by the incident tension, i.e.,

$$P_j = -\frac{a_j}{R_j} \frac{1}{\gamma(\xi_j) + a_j}, \quad j = 1 \text{ or } 2. \quad (78)$$

The scattered field p^s is again described by Eqs. (12) and (13), so that Eqs. (76) become

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} \left(1 + \frac{a_1}{\gamma} \right) S e^{ik_f \xi x} d\xi = -\frac{a_1}{R_1} \Gamma_R e^{-ik_f \xi_1 x}, \quad x < 0, \quad (79a)$$

$$\begin{aligned} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left(1 + \frac{a_2}{\gamma} \right) S e^{ik_f \xi x} d\xi \\ = -\frac{a_2}{R_2} \Gamma_T e^{ik_f \xi_2 x} - [\gamma(\xi_1) + a_2] P_1 e^{ik_f \xi_1 x}, \quad x > 0. \end{aligned} \quad (79b)$$

Assuming that ξ_0 , ξ_1 , and ξ_2 each have small positive imaginary parts, it can be easily checked that the solution to the dual integral equation is

$$\begin{aligned} \left(1 + \frac{a_2}{\gamma} \right) \frac{S}{K_+} = \frac{a_1}{R_1} \frac{\Gamma_R}{K_-(-\xi_1)} \frac{1}{\xi + \xi_1} \\ - \frac{a_2}{R_2} \frac{\Gamma_T}{K_+(\xi_2)} \frac{1}{\xi - \xi_2} \\ - [\gamma(\xi_1) + a_2] \frac{P_1}{K_+(\xi_1)} \frac{1}{\xi - \xi_1}. \end{aligned} \quad (80)$$

B. Acoustic diffraction and reciprocity

The scattered wave field p^s is determined from Eqs. (13), (77), and (80) by capturing the physically significant poles at $\xi = -\xi_1$ for $x < 0$ and $\xi = \xi_0, \xi_2$ for $x > 0$, thus

$$p^{(0)} = \begin{cases} P_1 e^{ik_f [\xi_1 x + \gamma(\xi_1)y]} + \Gamma_R P_1 e^{ik_f [-\xi_1 x + \gamma(\xi_1)y]} + p_d^s, & x < 0, \\ \Gamma_T P_2 e^{ik_f [\xi_2 x + \gamma(\xi_2)y]} + p_d^s, & x > 0, \end{cases} \quad (81)$$

where p_d^s describes that portion of the acoustic wave field diffracted from the joint at $x=0$. A diffraction coefficient $\bar{D}_1(\theta)$ can be defined for the scattered pressure, in a similar manner to Eq. (36). We find, after some simplification, that the far field in the direction θ depends upon

$$\bar{D}_1(\theta) = \frac{1}{4} p_0(\pi - \theta) \left\{ \frac{a_1}{R_1} \frac{K_+(-\cos \theta_1)}{\cos \theta - \cos \theta_1} + \frac{a_1}{R_1} \frac{\Gamma_R K_+(\cos \theta_1)}{\cos \theta + \cos \theta_1} - \frac{a_2}{R_2} \frac{\Gamma_T K_-(-\cos \theta_2)}{\cos \theta - \cos \theta_2} \right\}. \quad (82)$$

Finally, we note that a simple reciprocal relation exists between the diffraction coefficients D_1 and $\bar{D}_1$. The relation follows from the general principle of acoustic reciprocity, applied to the specific fluid-structure problem considered here, and is derived in Appendix C in the form

$$\bar{D}_1(\theta) = \frac{1}{2} k_f^2 a_1 \xi_1 D_1(\pi - \theta). \quad (83)$$

It can be easily checked that this identity is satisfied by D_1 and $\bar{D}_1$ of Eqs. (70a) and (82). A similar reciprocal identity exists for the diffraction coefficient D_2 and the corresponding coefficient $\bar{D}_2$ for membrane wave incidence from plate 2. These reciprocity identities are generally useful in that they remove the necessity to undertake separate analyses for the acoustic-to-membrane and the membrane-to-acoustic diffraction problems. We have done so here because the solutions are relatively simple in form, but in dealing with more sophisticated plate models one need only solve one or the other problem.

V. CONCLUSIONS

We have derived exact expressions for the acoustic diffraction from the junction of two massive plates. The diffraction coefficient D of Eq. (39) depends upon the near-field total pressure at the junction, p_0 of Eq. (32). These expressions are relatively simple in form, as compared with, for example, analogous expressions for plates with finite bending stiffness.² In particular, the frequency dependence of the diffraction coefficient D can be understood as a resonant-type transition from low-frequency to high-frequency behavior, corresponding to the pressure release and rigid limits of the boundary conditions, respectively. We have also derived approximate but simple expressions for the acoustic-to-membrane diffraction coefficients. These describe the tensile wave field in two joined curved plates. If one plate is flat then its diffracted tensile force is simply the total force at the junction. Using reciprocity, we have also obtained the membrane-to-acoustic diffraction coefficients for the pressure wave field generated in the fluid by an incoming membrane wave striking the joint of the two curved plates. The present results are part of a continued effort to derive the entire set of coupling coefficients necessary to describe the ray acoustics of complex structures. They offer, for the first time, the possibility of estimating the acoustic-to-membrane interactions at plate junctions.

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APPENDIX A: FACTORIZATION OF $K(\xi)$

To calculate the scattered wave field p^s given as Eq. (23) and the diffraction coefficient described by Eq. (38), one requires the knowledge of the split functions $K_-(\xi)$ and $K_+(\xi)$ defined by Eq. (18). Methods for obtaining the analytic factorization of a general kernel $K(\xi)$ are outlined by Noble.¹² It is useful to first consider the derivative of this function, $d \log K(\xi)/d\xi$. Alternatively, since $K(\xi)$ is the ratio of similar functions, we may write

$$\log K_-(\xi) = \log \eta_{1-}(\xi) - \log \eta_{2-}(\xi), \quad (A1)$$

where $\eta_{1,2}(\xi) = \gamma(\xi) + a_{1,2}$. Letting $\eta(\xi) \equiv \gamma(\xi) + a$, we are therefore led to consider the evaluation of

$$\frac{d}{d\xi} \log \eta_-(\xi) = \frac{-1}{2\pi i} \int_{-\infty}^{\infty} \frac{s}{\gamma(s)[\gamma(s) + a]} \frac{ds}{s - \xi}, \quad (A2)$$

where the contour along the real axis is indented over the pole at $s = \xi$ for real ξ .

The integral (A2) can be easily integrated by changing the integration variable according to $s = -\cosh t$, $\gamma(s) = -\sinh t$, and rearranging the contours of integration.¹³ A full discussion of this approach for kernels more general than that considered here will be included in a future paper by Norris and Wickham.⁴ After some straightforward manipulations, we find that

$$\frac{d}{d\xi} \log \eta_-(\xi) = \frac{1}{2(\xi - \alpha)} + \frac{i}{\pi(\alpha^2 - \xi^2)} \left(\alpha \cosh^{-1} \alpha - \frac{ia\xi}{\gamma(\xi)} \cos^{-1}(-\xi) \right), \quad (A3)$$

where

$$\alpha \equiv \sqrt{1 + a^2}. \quad (A4)$$

Noting that $\cos^{-1}(-\xi) = \pi/2 + \sin^{-1}(\xi)$, we may integrate (A3) to yield

$$\eta_-(\xi) = \sqrt{\alpha - \xi} \exp \left(\frac{i}{2} \tan^{-1} \frac{a}{i\gamma(\xi)} + iI(\xi) \right), \quad (A5)$$

where

$$I(\xi) = \frac{1}{\pi} \int_0^{\xi} \left(\alpha \cosh^{-1} \alpha - \frac{ia\xi}{\gamma(\xi)} \sin^{-1}(t) \right) \frac{dt}{\alpha^2 - t^2}. \quad (A6)$$

The constant of integration in (A5) has been chosen in accordance with the requirement that $\eta_-(0) = [\eta(0)]^{1/2}$, which is a consequence of the symmetry property $\eta(-\xi) = \eta_+(\xi)$ evaluated at $\xi=0$. Equations (19), (A1), and (A5) now imply that

$$K_-(\xi) = [K(\xi)]^{1/2} \left(\frac{(\alpha_1 - \xi)(\alpha_2 + \xi)}{(\alpha_1 + \xi)(\alpha_2 - \xi)} \right)^{1/4} \times \exp[iI_1(\xi) - iI_2(\xi)], \quad (\text{A7})$$

and the function $K_+(\xi)$ then follows from (19).

Another representation of the split functions of $K(\xi)$ can be obtained by using the change of variable $\xi = \cos \theta$. Then,

$$\eta_-(\xi) = \sqrt{\alpha - \xi} [\mathcal{R}(0)]^{-1/4} \exp[\Psi(\xi)], \quad (\text{A8})$$

and

$$\Psi(\xi) = \frac{1}{\pi} \int_{\pi/2}^{\cos^{-1} \xi} \left(\frac{\theta_0 \cos \theta_0 \sin \theta + (\pi - \theta) \sin \theta_0 \cos \theta}{\cos^2 \theta_0 - \cos^2 \theta} \right) \times d\theta, \quad (\text{A9})$$

where $\alpha = -\cos \theta_0$ which gives $a = -i \sin \theta_0$ using Eq. (A4). Once again the constant of integration in (A8) is chosen to satisfy $\eta_-(0) = [\eta(0)]^{1/2}$. Equations (19), (A1), and (A8) now imply that

$$K_-(\xi) = \sqrt{\frac{\alpha_1 - \xi}{\alpha_2 - \xi}} \left(\frac{\mathcal{R}_2(0)}{\mathcal{R}_1(0)} \right)^{1/4} \exp[\Psi_2(\xi) - \Psi_1(\xi)]. \quad (\text{A10})$$

The function $K_+(\xi)$ then follows from (19) as

$$K_+(\xi) = \sqrt{\frac{\alpha_2 + \xi}{\alpha_1 + \xi}} \left(\frac{\mathcal{R}_1(0)}{\mathcal{R}_2(0)} \right)^{1/4} \exp[\Phi_1(\xi) - \Phi_2(\xi)], \quad (\text{A11})$$

where

$$\Phi(\xi) = \frac{1}{\pi} \int_{\pi/2}^{\cos^{-1} \xi} \left(\frac{\theta \cos \theta \sin \theta_0 - \theta_0 \cos \theta_0 \sin \theta}{\cos^2 \theta_0 - \cos^2 \theta} \right) d\theta. \quad (\text{A12})$$

This form of K_+ is a limiting case of that derived by Norris and Wickham⁴ for two joined curved plates.

APPENDIX B: LOW AND HIGH FREQUENCY LIMITS FOR D

The low- and high-frequency forms for the diffraction coefficient in Eqs. (42) and (43), respectively, may be obtained by using suitable asymptotic approximations based on the coefficient $a_{1,2}$ of Eq. (4), and corresponding perturbation expansions of the pressure p . We discuss the two limits separately.

Assume that the frequency ω is much greater than both $\omega_{1,2}$ so that $a_{1,2} \ll 1$ and choose the following *ansatz* for the background pressure

$$p = p^0 + p^1 + \dots, \quad (\text{B1})$$

where $p^1 = O(a_{1,2})$. Then substitute into the leading-order form for the boundary conditions of Eq. (3), to obtain

$$\frac{\partial p^0}{\partial y} = 0, \quad -\infty < x < \infty, \quad y = 0. \quad (\text{B2})$$

The leading-order pressure therefore sees a rigid surface, and

$$p^0 = p^i + e^{ik_f r \cos(\theta - \theta_0)}, \quad (\text{B3})$$

such that the reflection coefficient is +1. The next term in the expansion gives, after substitution of Eq. (B3),

$$\frac{1}{k_f} \frac{\partial p^1}{\partial y}(x, 0) = 2e^{ik_f x \cos \theta_0} \begin{cases} a_1, & x < 0, \\ a_2, & x > 0, \end{cases} \quad (\text{B4})$$

and p^1 satisfies the Helmholtz equation (6) for $y \geq 0$. This boundary value problem can be solved by application of Green's theorem using the Green's function for a rigid boundary,

$$G_{\text{Rigid}}(x, y; x', y') = G_{\text{Free}}(x, y; x', y') + G_{\text{Free}}(x, y; x', -y'), \quad (\text{B5})$$

where the 2-D free space Green's function is $G_{\text{Free}} = (-i/4)H_0^{(1)}(k_f r)$. Then,

$$p^1(x, y) = \int_{-\infty}^{\infty} G_{\text{Rigid}} \frac{\partial p^1}{\partial y} dx', \quad y' = 0. \quad (\text{B6})$$

To obtain the diffraction coefficient, Eqs. (B5) and (B6), are approximated for large $k_f r \rightarrow \infty$ to obtain a form similar to Eq. (36). The diffraction coefficient arises from the discontinuity in the field $\partial p^1(x, 0)/\partial y$ at $x = 0$, which yields end point contributions at $x = \pm 0$. The manipulations are straightforward with the result identical to that obtained in Eq. (44).

In the low-frequency limit ω is much smaller than both $\omega_{1,2}$, so that $a_{1,2} \gg 1$. The same *ansatz* (B1) is applied but now with $p^1 = O(1/a_{1,2})$, to obtain

$$p^0 = 0, \quad -\infty < x < \infty, \quad y = 0. \quad (\text{B7})$$

The leading-order solution satisfies a pressure-release boundary condition, and

$$p^0 = p^i - e^{ik_f r \cos(\theta - \theta_0)}. \quad (\text{B8})$$

The next term in the expansion gives, after substitution of Eq. (B8),

$$p^1(x, 0) = -2i \sin \theta_0 e^{ik_f x \cos \theta_0} \begin{cases} a_1^{-1}, & x < 0, \\ a_2^{-1}, & x > 0, \end{cases} \quad (\text{B9})$$

supplemented by Eq. (6) for p^1 . Once again this can be solved by application of Green's theorem, but now the pressure release Green's function is used. Proceeding as before, approximating for large $k_f r \rightarrow \infty$ we find that the diffraction coefficient reduces to Eq. (42).

APPENDIX C: RECIPROCITY FOR THE COUPLING COEFFICIENTS

A general statement of acoustic reciprocity for fluid-structure interaction was recently given by Norris and Rebinsky.¹⁴ They showed that the displacement in direction $\mathbf{e}^{(2)}$ at any position $\mathbf{x}^{(2)}$ in the structure due to a monopole of strength $(\rho_f \omega^2)$ at arbitrary position $\mathbf{x}^{(1)}$ in the fluid is the same as the pressure at $\mathbf{x}^{(1)}$ caused by a unit force in the direction $\mathbf{e}^{(2)}$ applied at $\mathbf{x}^{(2)}$. Let $\mathbf{u}^{(1)}$ be the structural displacement resulting from the monopole of strength $\rho_f \omega^2$, and $p^{(2)}$ the acoustic pressure caused by the force on the structure, then

$$\mathbf{e}^{(2)} \cdot \mathbf{u}^{(1)}(\mathbf{x}^{(2)}) = p^{(2)}(\mathbf{x}^{(1)}). \quad (\text{C1})$$

For the problem at hand we consider a point force on plate 1 in the positive x direction, $e^{(2)}$, at position $x = x^{(2)}$ where $x^{(2)} < 0$ by assumption. In keeping with the leading-order decoupling between the fluid and the membrane waves, the stress satisfies [cf. Eq. (45a)]

$$\tau_{,x}^{(2)} + m_1 \omega^2 v^{(2)} = \delta(x - x^{(2)}), \quad (C2)$$

with solution $\tau^{(2)}(x) = \pm \frac{1}{2} \exp(ik_1|x - x^{(2)}|)$ for $x \gtrless x^{(2)}$. Let the point in the fluid, $\mathbf{x}^{(1)}$, lie in direction θ in the far field, so that we can then use the membrane-to-acoustic diffraction coefficient to evaluate the acoustic response from the junction. Using the definition of $\tilde{D}_1$ and Eq. (36), we have

$$p^{(2)}(\mathbf{x}^{(1)}) = \frac{1}{2} \tilde{D}_1(\theta) \sqrt{\frac{2}{\pi k_f r^{(1)}}} e^{-i\pi/4} e^{ik_f r^{(1)}} e^{ik_1|x^{(2)}|}. \quad (C3)$$

At the same time, the monopole in the fluid induces an incident pressure $p^{(1)}(0,0) = (\rho_f \omega^2)(-i/4)H_0^{(1)}(k_f r^{(1)})$ at the junction. This acoustic field produces diffracted membrane stresses in both plates, where the applicable angle of incidence is $\pi - \theta$. The associated in-plane displacement in plate 1 can be found from Eq. (65) as

$$v^{(1)}(x^{(2)}) = \frac{ik_1}{m_1 \omega^2} D_1(\pi - \theta) p^{(1)}(0,0) e^{ik_1|x^{(2)}|}, \quad (C4)$$

which is precisely the left member in Eq. (C1). Therefore, equating the right members in Eqs. (C3) and (C4) and ap-

proximating the Hankel function for large argument, we deduce the reciprocal identity (83).

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