Nonlinear tube waves

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(Received 31 December 1993; accepted for publication 25 April 1994)

The nonlinear characteristics of an acoustic tube wave propagating along the axis of a fluid-filled circular borehole in an elastic solid that is locally isotropic but whose properties may vary radially is considered. The analysis is carried out in the quasistatic limit. All terms through quadratic in the amplitude of the wave are considered and the amplitude of second-harmonic generation and the pressure dependence of the tube wave speed, dV_T/dp, are expressed in terms of the fluid and formation nonlinear parameters. The results show that if there is no radial variation of the shear modulus of the solid then both the amplitude of second-harmonic generation and dV_T/dp are independent of the third-order elastic constants of the solid and nearly equal to those of the fluid alone. If there is a radial variation of the shear modulus then the numerical calculations indicate that both the amplitude of second-harmonic generation and dV_T/dp can be completely dominated by the nonlinear parameters of the solid. A perturbation theory valid for the case in which the shear modulus is nearly constant is derived that demonstrates that the nonlinear response is scaled by the value of the third-order parameters of the solid, leveraged by the degree and depth of alteration of the shear modulus.

PACS numbers: 43.25.Ts

INTRODUCTION

In the last 10–15 years the acoustic properties of a fluid-filled cylindrical cavity in an elastic solid have received a great deal of attention, not least because of its technical importance in the search for oil reserves. The classic papers by Tsang and Rader[1] and Kurkjian and Chang[2] have established that, depending on the spatial dependence of the sourcing transducer, the acoustic signal is composed of many different components. Some of these are normal modes of the system, corresponding to poles in the appropriate Green's function, and others, "head waves," corresponding to branch points. If the source has an axially symmetric component, it is known that at low frequencies the dominant contribution to the acoustic signal is due to the lowest-lying mode in the problem, the "tube wave." Here, low frequency means that the wavelengths of all bulk modes are large compared to the borehole radius. In this limit the tube wave becomes nondispersive with a speed V_T, which is simply expressed in terms of the properties of the fluid and the solid [Eq. (11)]; moreover, the spatial dependences of the pressure in the fluid and the displacement in the solid also become simple. In this paper we consider the nonlinear properties of such a system under the assumption that the linear properties are dominated by a low-frequency tube wave. We develop an equation of motion for the pressure in the borehole fluid that we use to calculate both the change in tube wave speed with pressure, dV_T/dp, and the amplitude of a second-harmonic tube wave generated by the nonlinearities of both fluid and solid. We limit our analysis to nonlinearities quadratic in the wave amplitude only.

The theoretical and experimental connection between nonlinearity and second-harmonic generation is well established for bulk plane waves in both fluids and solids and even for certain surface waves at planar interfaces.[3–9] Here we consider the more difficult mathematics appropriate to the cylindrical geometry of the borehole because sedimentary rocks can have enormously large values of the so-called third-order elastic constants. We show that second-harmonic generation and the pressure dependence of speed should be relatively easy to observe and therefrom deduce information about the rock's nonlinearity. As a thumbnail demonstration of this claim we consider how the speed of longitudinal sound V_P and transverse sound V_S change with the application of a confining pressure. We have found it convenient to use the Landau and Lifshitz[10] convention for the third-order constants A, B, and C. In terms of these parameters one has[11–13]

$$\rho_0 \frac{dV_S^2}{dP} = -\frac{3\lambda+6\mu+A+3B}{3\lambda+2\mu}, \tag{1}$$

$$\rho_0 \frac{dV_P^2}{dP} = -\frac{7\lambda+10\mu+2A+10B+6C}{3\lambda+2\mu}, \tag{2}$$

where λ and μ are the Lamé constants of the solid, and ρ_0 is its density in the undeformed state. In Table I we list some values of these dimensionless groups for a variety of materials.[11,14–21]

In Fig. 1(a) and (b) we plot the speeds of sound for a dry Boise Sandstone[22] as a function of pressure. The solid lines are simply smooth curves constructed for the purposes of differentiation according to (1) and (2). The results are plotted in Fig. 2(a) and (b) where we see that for this rock the size of the nonlinear parameters relative to the linear can be

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TABLE I. Dimensionless dependence of sound speed on pressure for some common materials.

Material	$\rho_0\, dV_S^2/dP$	$\rho_0\, dV_P^2/dP$
Water[a]	0	5.0
Benzene[a]	0	9.0
Polystyrene[b]	1.57	11.6
PMMA[c]	3.0	15.0
Pyrex[b]	−2.84	−8.6
Fused silica[d]	−1.42	−4.32
Alumina[e]	1.12	4.46
Aluminum[f]	2.92	12.4
Nickel–steel[g]	1.55	2.84
Armco–Iron[h]	5.7	9.3
Steel (Hecla)[f]	1.46	7.45
Molybdenum[f]	1.05	3.48
Tungsten[f]	0.70	4.58
Magnesium[f]	1.47	6.89
Niobium[h]	0.29	6.18
Gold[i]	0.90	6.4

[a]Reference 14.
[b]Reference 11.
[c]Reference 15.
[d]Reference 16.
[e]Reference 17.
[f]Reference 18.
[g]Reference 19.
[h]Reference 20.
[i]Reference 21.

orders of magnitude larger than for the materials listed in Table I. Indeed nonlinear phenomena seem to be easily observed in rocks.[23] Our intuition based on Fig. 2 as compared with Table I is that a tube wave propagating in a sedimentary rock/borehole environment can exhibit significant nonlinear effects due mostly to those of the rock. We make this claim concrete in the remainder of the paper.

Conceptually, the idea of second-harmonic generation of a tube wave is simple. It is mathematically complex because one is dealing with nonlinearities of a combined fluid and solid system, in cylindrical coordinates, and (as we show below) it is necessary to consider spatially varying elastic constants. The outline of the paper is as follows: In Sec. I we develop an approximate nonlinear wave equation for the pressure in a borehole, which is valid when the quantities are slowly varying and when the nonlinearities are weak effects compared to the linear; we consider equations of motion that are accurate through second order in the wave amplitude only. We relate the amplitude of second-harmonic generation to the linear and nonlinear properties of the borehole fluid and to the linear and nonlinear static response of the borehole wall to an applied pressure. In like manner we derive the expression relating the initial change in tube wave speed to an applied pressure, $dV_T/dp|_{p=0}$. In Sec. II we calculate the borehole response using standard second-order elasticity theory applied to the cylindrical borehole problem. We show that if the shear modulus of the solid is spatially uniform, then, regardless of the spatial dependence of the other moduli, the linear response is easily given in terms of that shear modulus and the quadratic dependence of the borehole expansion on (Lagrangian) pressure is identically zero. We numerically investigate cases in which the shear modulus varies radially and we show that the quadratic dependence is enhanced if the radial depth of shear alteration is comparable to the borehole radius. We derive a simple but effective perturbation theory valid when the shear modulus is almost constant. This theory illustrates the most important combination of nonlinear parameters; we propose a simple gedanken experiment whereby one could measure that combination directly. Our concluding remarks are presented in Sec. III where we use values of available material parameters to estimate the efficiency of second-harmonic generation as well as dV_T/dp.

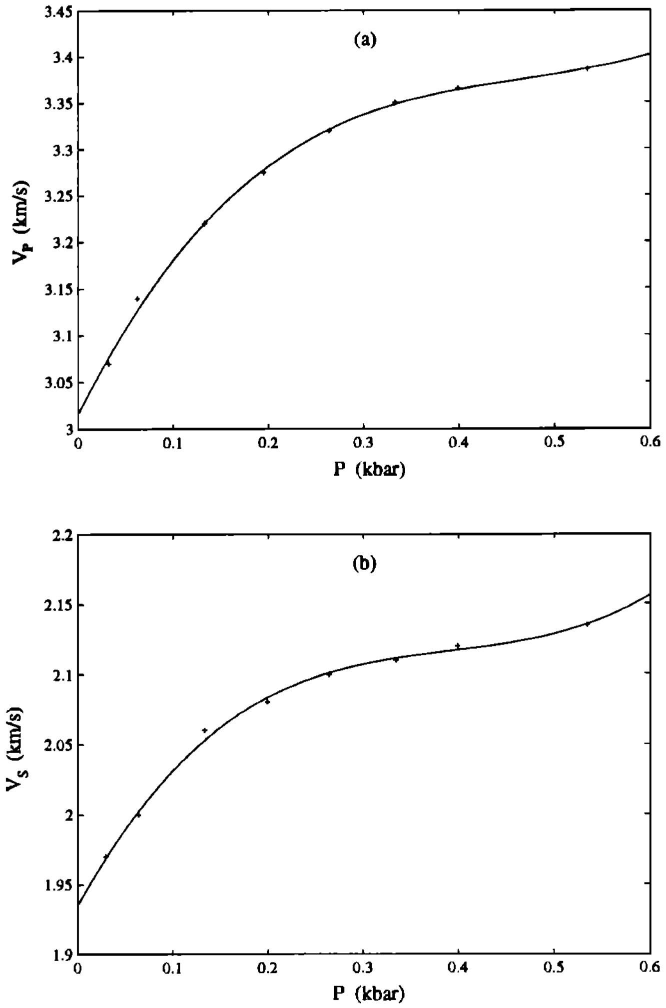

FIG. 1. (a) Longitudinal and (b) transverse speed of sound for a dry Boise Sandstone as a function of pressure (from Ref. 22). The solid curves are simple analytic functions constructed to fit the data.

I. QUASISTATIC LIMIT OF A NONLINEAR TUBE WAVE

Here we consider the relevant wave equation for the pressure in the borehole. We carry out the analysis in the quasistatic limit and we consider all terms and only those terms that are either linear or quadratic in the wave amplitude. Our spirit is much like that of White's[24] treatment of the linear problem.

We consider an infinitely long, fluid-filled cylinder of radius a in an elastic solid. The coordinate origin is at the center of the borehole. The properties of the solid may vary radially but not axially or azimuthally. It shall prove to be convenient to work with Eulerian (current position) variables for the fluid, as is customary, but Lagrangian (original position) variables for the solid, as is also customary: The *current*

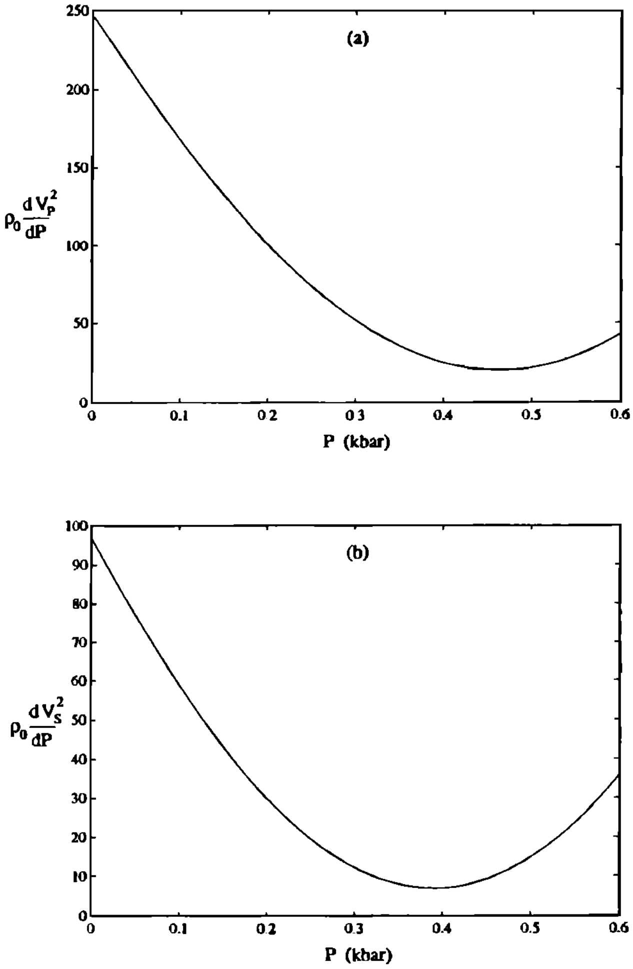

FIG. 2. The dimensionless quantities $\rho_0\, dV^2/dP$ for (a) longitudinal and (b) transverse speed of sound in the Boise Sandstone of Fig. 1. The derivative was taken from the analytic curves in Fig. 1.

position of a material particle in the fluid is labeled by (r,θ,z) whereas the *original* position of a solid particle in the formation is also labeled by (r,θ,z). In the quasistatic limit one may safely neglect the radial component of the fluid motion (it is smaller than the axial by a factor $\propto\omega^1$) and we may neglect any radial variation of p, ρ, and v for the same reason. Whitham[25] considers similar approximations for a similar problem involving shock dynamics in a tube of variable cross-sectional area. Thus $p(z,t)$, $\rho(z,t)$, and $v(z,t)$ are, respectively, the actual pressure, density, and axial fluid velocity in the borehole at position z at time t, but $\mathbf{u}(\mathbf{X},t)$ is the displacement of a material particle in the solid that was originally at $\mathbf{X}$ and is now at position $\mathbf{X}+\mathbf{u}(\mathbf{X},t)$. In this section we shall need only the radial displacement at the borehole wall, which we denote as $u(z,t)\equiv u_r\ (r=a,z,t)$ in the notation of the Appendix.

We have the equation of motion for the fluid,

$$\rho(v_{,t}+vv_{,z})+p_{,z}=0, \tag{3}$$

where the comma followed by a variable denotes differentiation with respect to that variable. The radial motion of the borehole wall does not enter this equation. We consider the following equation of state for the fluid:[26]

$$p=A_f\left(\frac{\rho-\rho_0}{\rho_0}\right)+\frac{1}{2}B_f\left(\frac{\rho-\rho_0}{\rho_0}\right)^2+\cdots, \tag{4}$$

where p is the acoustic pressure (relative to ambient), and ρ_0 is the ambient density. The constant A_f is the bulk modulus of the fluid and so the speed of small-amplitude sound waves in the fluid is $c=(A_f/\rho_0)^{1/2}$. We need (4) in inverted form:

$$\rho=\rho_0+\frac{1}{c^2}p-\frac{B_f}{2c^2A_f^2}p^2+\cdots. \tag{5}$$

Thus Eq. (3) can be written, correct to $O(p^2)$, as

$$\rho_0 v_{,t}+p_{,z}+(1/c^2)pv_{,t}+\rho_0 vv_{,z}=0. \tag{6}$$

Consider, now, the continuity equation for the fluid entrained between the planes z and $z+\Delta z$:

$$\int \mathbf{j}\cdot\hat{\mathbf{n}}\,dA+\frac{dM}{dt}=0, \tag{7}$$

where $\mathbf{j}$ is the mass flux and M is the mass. For the case at hand we have

$$\pi[a+u(z+\Delta z)]^2\rho(z+\Delta z)v(z+\Delta z)-\pi[a+u(z)]^2 \times\rho(z)v(z)+\frac{\partial}{\partial t}\{\pi[a+u(z)]^2\rho(z)\Delta z\}=0, \tag{8}$$

where the time derivative is indeed a partial, not a convective, derivative. In order to eliminate $u(z,t)$ from (8) we need the response of the borehole wall to a static pressure:

$$u=(aN_1/2)(p+N_2p^2+\cdots), \tag{9}$$

where N_1 and N_2 are the linear and nonlinear compliances, respectively. We discuss this response of the borehole wall to pressure in Sec. II for a variety of cases of interest. Equation (8) is, to second order in the pressure, given by

$$\rho_0 v_{,z}+\frac{1}{V_T^2}p_{,t}+\frac{1}{V_T^2}(vp_{,z}+pv_{,z}) +\left(\frac{2N_1}{c^2}+\frac{\rho_0N_1^2}{2}+2\rho_0N_1N_2-\frac{B_f}{c^2A_f^2}\right)pp_{,t}=0. \tag{10}$$

where V_T is the small amplitude tube wave speed given by[24]

$$\frac{1}{V_T^2}=\rho_0\left(\frac{1}{A_f}+N_1\right)=\frac{1}{c^2}+\rho_0N_1. \tag{11}$$

We are now in a position to relate dV_T/dp as well as the amplitude of second-harmonic generation to the fluid parameters and the formation parameters, N_1 and N_2. They are slightly different problems and need to be treated separately.

A. Pressure dependence of tube wave speed

We consider that the pressure has a constant part p_0 as well as a small-amplitude wave part $\tilde{p}(z,t)$ but the velocity has only a small-amplitude part:

$$p=p_0+\tilde{p}(z,t),$$
$$v=\tilde{v}(z,t). \tag{12}$$

We consider (6) to first order in $\tilde{p}$ and $\tilde{v}$:

$$\left(\rho_0+\frac{p_0}{c^2}\right)\tilde{v}_{,t}+\tilde{p}_{,z}=0. \tag{13}$$

similarly for (10),

$$\left(\rho_0+\frac{p_0}{V_T^2}\right)\tilde{v}_{,z}+\left[\frac{1}{V_T^2}+\left(\frac{2N_1}{c^2}+\frac{\rho_0 N_1^2}{2}+2\rho_0 N_1 N_2 -\frac{B_f}{c^2 A_f}\right)p_0\right]\tilde{p}_{,t}=0. \tag{14}$$

It is straightforward to eliminate $\tilde{v}$ from (13) and (14),

$$V_T^2(p_0)\tilde{p}_{,zz}-\tilde{p}_{,tt}=0, \tag{15}$$

and obtain an expression for the tube wave speed $V_T(p_0)$ accurate to first order in p_0. This dependence may most conveniently be expressed by analogy with Eq. (1):

$$\rho_0 \left.\frac{dV_T^2}{dp_0}\right|_{p_0=0}=\frac{B_f/A_f-A_fN_1+A_f^2N_1^2/2-2A_f^2N_1N_2}{(1+A_fN_1)^2}. \tag{16}$$

We see that in the special case that the formation is rigid, $N_1\rightarrow 0$, we have

$$\rho_0 \left.\frac{dc^2}{dp_0}\right|_{p_0=0}=\frac{B_f}{A_f}, \tag{17}$$

as has been known since antiquity.[27] Equation (16) is the low-frequency limit of the dispersive tube wave considered by Sinha *et al.*[28]

B. Second-harmonic generation

We now turn our attention to the efficiency with which a monochromatic wave of frequency ω can generate a second-harmonic wave of frequency 2ω. First, we eliminate v from all the quadratic terms in (6) and (10) only. We do this by using the relationships implied by the first-order equations:

$$v_{,t}=-(1/\rho_0)p_{,z} \tag{18}$$

and

$$v_{,z}=-(1/\rho_0 V_T^2)p_{,t}. \tag{19}$$

Moreover, we assume that we are dealing with waves that are primarily traveling up or down the borehole. Either (18) or (19) implies

$$v=\pm(1/\rho_0 V_T)p \tag{20}$$

and, of course,

$$\frac{\partial}{\partial t}=\mp V_T\frac{\partial}{\partial z}. \tag{21}$$

With these approximations to the quadratic terms Eqs. (6) and (10) become

$$\rho_0 v_{,t}+p_{,z}+N_1 p p_{,z}=0 \tag{22}$$

and

$$\rho_0 v_{,z}+\frac{1}{V_T^2}p_{,t}+\left(\frac{N_1}{V_T^2}-\alpha\right)pp_{,t}=0, \tag{23}$$

respectively, where

$$\alpha=\rho_0\left[\frac{1}{A_f^2}\left(2+\frac{B_f}{A_f}\right)+\frac{3N_1}{A_f}+N_1\left(\frac{5}{2}N_1-2N_2\right)\right]. \tag{24}$$

Combining these two equations we have

$$p_{,zz}-(1/V_T^2)p_{,tt}+\alpha(pp_{,t})_{,t}=0. \tag{25}$$

An equation of this form has been derived by Westervelt[29] for the case of a homogeneous infinite fluid medium. The parameter α governs the degree of nonlinearity at this level of approximation. The first two terms of (24) are exactly that combination of fluid parameters that determines the amplitude of second-harmonic generation of plane waves in a fluid.[27] Indeed, if the borehole wall is rigid ($N_1\rightarrow 0$) we recover this result, as seems intuitively obvious. The effects of the solid's linear and nonlinear constitutive behaviour are folded into the parameters N_1 and N_2, which we consider (in a different notation) in Sec. II.

In the rest of this section we consider the amplitude of second-harmonic generation implied by (25). We consider first the case in which a radially independent source generates a monochromatic signal at $z=0$ that radiates both up and down the borehole. The nonlinearity in (25) generates a second harmonic that also radiates away from the source. We follow the usual practice of three-wave mixing theory[25] and assume that the amplitude of the second harmonic is slowly varying as a function of position:

$$p(z,t)=P_0\cos(k|z|-\omega t)+f(z)\sin(2k|z|-2\omega t)+g(z)\cos(2k|z|-2\omega t), \tag{26}$$

where $k=\omega/V_T$. For our purposes we may neglect the z dependence of P_0. We substitute (26) into (25) and we neglect second derivatives of f and g. Consistency with (25) requires

$$\frac{dg}{dz}=0, \tag{27}$$

$$\frac{df}{dz}=\pm\frac{\alpha V_T\omega P_0^2}{4}, \tag{28}$$

where the $\pm$ refers to $z>0$ or $z<0$. Thus

$$p(z,t)=P_0\cos(k|z|-\omega t)+\left(\frac{\alpha V_T\omega P_0^2}{4}|z|+f_0\right)\times\sin(2k|z|-2\omega t)+g_0\cos(2k|z|-2\omega t), \tag{29}$$

where f_0 and g_0 are constants. It is often assumed in standard texts that the second harmonic "starts" at $z=0$, implying that these constants are zero. We believe that this is almost never the correct boundary condition. In the case at hand we require that the second-harmonic component of the fluid velocity be continuous; the fundamental component is discontinuous due to the monochromatic source. Thus the second-harmonic component of $v_{,t}$ must also be continuous. Substituting Eq. (29) into (22) and imposing continuity of the 2ω component of $v_{,t}$ at $z=0$ implies

$$f_0=0, \tag{30}$$

$$g_0=[(\alpha V_T^2-2N_1)/8]P_0^2. \quad (31)$$

Equation (29) now becomes

$$p(z,t)=P_0\cos(k|z|-\omega t)+P_0^2\left(\frac{\alpha V_T\omega}{4}|z|\sin(2k|z|-2\omega t)+\frac{\alpha V_T^2-2N_1}{8}\cos(2k|z|-2\omega t)\right). \quad (32)$$

We also have

$$v=\pm\frac{1}{\rho_0 V_T}P_0\cos(k|z|-\omega t)+\frac{\alpha\omega}{4\rho_0}P_0^2 z\times\sin(2k|z|-2\omega t). \quad (33)$$

We note in passing that continuity of $u(r)$ at $z=0$ is assured for all r regardless of the choice of f_0 and g_0. The ratio of the two second-harmonic terms is $\approx 2|z|\omega/V_T$. Thus, according to (32) the amplitude of the second harmonic is approximately independent of z within one wavelength of the source and thereafter increases linearly, as is common for problems of this sort. This perturbation theory is not valid once the amplitude becomes comparable to that of the fundamental for then the latter must decrease as a consequence of energy conservation. Also higher harmonics are generated. These points are discussed by, for example, Beyer and Letcher.[27]

If the source is not radially independent of position, then, within the context of linear theory, it generates a complex signal involving head waves, higher-order modes evanescent below cutoff, etc. These other components decay rapidly and so we may still expect (32) to hold for $z>a$.

II. NONLINEAR BOREHOLE COMPLIANCE

In this section we consider the static response of an elastic solid through which a cylindrical hole has been drilled. The solid is presumed to extend to infinity; in this axisymmetric problem the elastic parameters may be radially dependent on position. The borehole pressure is uniform. We wish to calculate the radial displacement in the formation $u(r)$ to second order in the pressure. For this purpose it is most convenient to use Lagrangian variables. Here, the Lagrangian pressure P_L is defined such that P_L times the original area of some surface element is the actual force on that element, though the area may be changed because of the deformation; it is related to the more conventional Eulerian pressure P_E by

$$aP_L=[a+u(a)]P_E, \quad (34)$$

where $u(r)$ $(=u_r$ in the notation of the Appendix) is the radial displacement of a point originally at a radial position r, and a is the unperturbed borehole radius. In Sec. I we used the symbol p to represent the conventional Eulerian pressure, P_E; in this section we use the symbol P for the Lagrangian pressure, P_L. In Sec. III we connect the results of Secs. I and II.

The Piola–Kirchhoff stress tensor $\mathbf{T}$ in an axisymmetric problem is given, to second order in the displacement, by Eqs. (A14)–(A18) of the Appendix. (It is not necessary to read the Appendix at this point, but we do quote certain results therefrom.) For this statics problem we have, from (A7),

$$\nabla\cdot\mathbf{T}(\mathbf{u})=0, \quad (35)$$

subject to the boundary condition on the (unperturbed) borehole wall

$$\mathbf{T}(\mathbf{u})\cdot\hat{\mathbf{n}}=-P\hat{\mathbf{n}} \quad \text{at } r=a, \quad (36)$$

where $\hat{\mathbf{n}}$ is a unit vector normal to the borehole wall. In this paper we are concerned only with a single well in an unbounded formation. Therefore one has the boundary condition that the stresses vanish at infinity

$$\lim_{r\to\infty}\mathbf{T}=\mathbf{0}. \quad (37)$$

To solve these equations we expand the solution in powers of P as

$$\mathbf{u}=P\mathbf{u}^{(1)}+P^2\mathbf{u}^{(2)}+\cdots. \quad (38)$$

The stress tensor can be decomposed into linear and nonlinear parts:

$$\mathbf{T}(\mathbf{u})=\mathbf{T}^{\mathrm{L}}(\mathbf{u})+\mathbf{T}^{\mathrm{NL}}(\mathbf{u}). \quad (39)$$

Because of (38), (39) can be written as

$$\mathbf{T}(\mathbf{u})=P\mathbf{T}^{\mathrm{L}}(\mathbf{u}^{(1)})+P^2[\mathbf{T}^{\mathrm{L}}(\mathbf{u}^{(2)})+\mathbf{T}^{\mathrm{NL}}(\mathbf{u}^{(1)})]+\cdots. \quad (40)$$

Equation (35) must hold order by order in P. To first order in P this is

$$\nabla\cdot\mathbf{T}^{\mathrm{L}}(\mathbf{u}^{(1)})=0. \quad (41)$$

Similarly, to first order in P, the boundary condition (36) is

$$P\mathbf{T}^{\mathrm{L}}(\mathbf{u}^{(1)})\cdot\hat{\mathbf{n}}=-P\hat{\mathbf{n}} \quad \text{at } r=a. \quad (42)$$

These are, of course, the ordinary equations of standard linear elasticity for which $u^{(1)}$ can be solved by standard means, as we illustrate below.

Similarly, the second-order solution is determined by the requirement that (35) hold to second order in P. From (40) we have

$$\nabla\cdot\mathbf{T}^{\mathrm{L}}(\mathbf{u}^{(2)})=-\nabla\cdot\mathbf{T}^{\mathrm{NL}}(\mathbf{u}^{(1)}), \quad (43)$$

with the boundary condition that follows from (36) and (40)

$$\mathbf{T}^{\mathrm{L}}(\mathbf{u}^{(2)})\cdot\hat{\mathbf{n}}=-\mathbf{T}^{\mathrm{NL}}(\mathbf{u}^{(1)})\cdot\hat{\mathbf{n}} \quad \text{at } r=a. \quad (44)$$

Thus we see that, having solved the linear problem for $\mathbf{u}^{(1)}$, we use it as an input for the sourcing of the second-order term, $\mathbf{u}^{(2)}$, in the differential equation (43), as well as in the boundary condition (44).

For the axisymmetric case at hand, for which there is only a radial displacement, we have from the Appendix

$$T_{rr}^{\mathrm{L}}(u)=(\lambda+2\mu)\frac{du}{dr}+\lambda\frac{u}{r}, \quad (45)$$

$$T_{\theta\theta}^{\mathrm{L}}(u)=\lambda\frac{du}{dr}+(\lambda+2\mu)\frac{u}{r}, \quad (46)$$

$$T^{\rm NL}_{rr}(u)=\left(\frac{3}{2}\lambda+3\mu+A+3B+C\right)\left(\frac{du}{dr}\right)^2+(\lambda+2B+2C)\times\frac{u}{r}\frac{du}{dr}+\left(\frac{\lambda}{2}+B+C\right)\left(\frac{u}{r}\right)^2. \tag{47}$$

We will also need the combination

$$T^{\rm NL}_{rr}(u)-T^{\rm NL}_{\theta\theta}(u)=(\lambda+3\mu+A+2B)\left[\left(\frac{du}{dr}\right)^2-\left(\frac{u}{r}\right)^2\right]. \tag{48}$$

These are the only relevant components of the stress tensor. The equilibrium equation (35) in cylindrical coordinates is

$$\frac{dT_{rr}}{dr}+\frac{1}{r}(T_{rr}-T_{\theta\theta})=0, \tag{49}$$

as is obvious from Eq. (A19). In Eqs. (41)–(49) the elastic constants λ, μ, A, B, and C may be presumed to be functions of radial position r. Therefore, Eq. (41) becomes

$$\frac{d}{dr}\left[\lambda(r)\left(\frac{du^{(1)}}{dr}+\frac{u^{(1)}}{r}\right)+2\mu(r)\frac{du^{(1)}}{dr}\right]+\frac{2\mu(r)}{r}\left(\frac{du^{(1)}}{dr}-\frac{u^{(1)}}{r}\right)=0, \tag{50}$$

and the boundary condition (42) is

$$T^{\rm L}_{rr}(u^{(1)})=\lambda(r)\left(\frac{du^{(1)}}{dr}+\frac{u^{(1)}}{r}\right)+2\mu(r)\frac{du^{(1)}}{dr}=-1 \quad \text{at } r=a. \tag{51}$$

Similarly, the second-order function $u^{(2)}(r)$ is determined from a solution to Eq. (43), which now reads

$$\frac{d}{dr}\left[\lambda(r)\left(\frac{du^{(2)}}{dr}+\frac{u^{(2)}}{r}\right)+2\mu(r)\frac{du^{(2)}}{dr}\right]+\frac{2\mu(r)}{r}\times\left(\frac{du^{(2)}}{dr}-\frac{u^{(2)}}{r}\right)=-\frac{d}{dr}T^{\rm NL}_{rr}(u^{(1)})-\frac{1}{r}[T^{\rm NL}_{rr}(u^{(1)})-T^{\rm NL}_{\theta\theta}(u^{(1)})], \tag{52}$$

subject to the boundary condition (44):

$$T^{\rm NL}_{rr}(u^{(1)})+T^{\rm L}_{rr}(u^{(2)})=0 \quad \text{at } r=a. \tag{53}$$

The displacement $u^{(2)}(r)$ satisfies an inhomogeneous, linear differential equation. As usual we may find any particular solution and add to it homogeneous solutions so as to satisfy the boundary conditions at the borehole wall and at infinity. For the purposes of second-harmonic generation of a tube wave we wish to know the displacement at the borehole wall, which we write in the form

$$u(a)=\frac{Pa}{2\mu^*}-\chi\frac{aP^2}{\mu_\infty^2}+O(P^3), \tag{54}$$

where $\mu_\infty=\mu(r\to\infty)$. As defined μ^* is an effective shear modulus and χ (dimensionless) will turn out to be non-negative for the cases in which we are interested. In the following subsections we consider solutions to this problem in increasing order of complexity.

A. Elastic moduli, λ, μ, A, B, and C independent of position

This is a situation for which the linear problem is treated in many textbooks. There are two linearly independent solutions to (50). If, in some region, μ is constant, then, regardless of $\lambda(r)$, one of the solutions is $u^{(1)}\propto r^{-1}$, which corresponds to a purely shear deformation with no volumetric change. In the special case that $\lambda(r)+\mu(r)$ is constant, the other solution is $u^{(1)}\propto r^{+1}$. This solution is purely compressive and, indeed, $\lambda+\mu$ is the two-dimensional (2-D) compressive modulus.

In the case at hand for which all the moduli are constant even as $r\to\infty$, the second solution violates the condition (37). Therefore, from (51) we have, for constant μ,

$$u^{(1)}(r)=a^2/2\mu r, \tag{55}$$

which implies

$$\mu^*=\mu. \tag{56}$$

From (47) and (48) we have

$$T^{\rm NL}_{rr}(u^{(1)})=T^{\rm NL}_{\theta\theta}(u^{(1)})=(\lambda+3\mu+A+2B)a^4/4\mu^2r^4. \tag{57}$$

There is a particular solution to Eq. (52) that is a power law, to which a multiple of (55) must be added so as to ensure the boundary condition (53). The result is

$$u^{(2)}(r)=-\frac{(\lambda+3\mu+A+2B)a^4}{8(\lambda+2\mu)\mu^2}\left(\frac{-1}{r^3}+\frac{1}{a^2r}\right). \tag{58}$$

This solution has the feature that there is no second-order contribution to the borehole stiffness; $u^{(2)}(a)=0$. From (38) and (54) we have

$$\chi\equiv 0. \tag{59}$$

For this case there is no contribution to the generation of a second harmonic from the third-order elasticity of the solid.

This exact cancellation at the borehole wall is an artifact, not only of the cylindrical geometry, but also of the fact that the formation extends to infinity; one can show that $u^{(2)}(a)\neq 0$ for the equivalent problem of a pipe with a finite outer radius, for example. We are led to consider the more general problem of arbitrary radial variation of the elastic parameters, of which a pipe with a finite outer radius is a special case.

B. Arbitrary $\lambda(r)$, $A(r)$, $B(r)$, and $C(r)$ but constant μ

The simplest example of a radially varying elastic formation is one in which $\mu(r)$ is constant in the entire interval $a<r<\infty$. The solution given by (55) still holds regardless of $\lambda(r)$. Therefore, $\mu^*\equiv\mu$ again. Equation (57) also holds with the understanding that $\lambda(r)$, $A(r)$, and $B(r)$ are arbitrary functions. The solution to (52) that satisfies the boundary condition (53) is

$$u^{(2)}(r)=-\frac{a^4}{4\mu^2 r}\int_a^r\frac{[\lambda(r)+3\mu+A(r)+2B(r)]}{[\lambda(r)+2\mu]r^3}\,dr. \tag{60}$$

[In the special case that λ, A, and B are constant, (60) reduces to (58).] Thus, as long as the shear modulus is constant we see that $u^{(2)}(a)=0 \Leftrightarrow \chi=0$. Therefore we are led to consider the case of radial variations in the shear modulus, $\mu(r)$.

C. Radial variation in the shear modulus, $\mu(r)$

If the shear modulus varies with position, it is not possible in general to reduce to quadrature a solution even of the linear problem (50). This is because at each point there are both compression and shear deformations. [Special cases occur when the solid is either incompressible, $\lambda\rightarrow\infty$, or perfectly compressible, $\lambda(r)+\mu(r)\equiv 0$, which cannot hold if the solid is isotropic in three dimensions.] Accordingly we have undertaken to solve Eqs. (50) and (52) numerically. Each can be put in the form

$$\frac{dy_i}{dr}=A_{ij}(r)y_j(r)+B_i(r),\quad i=1,2. \tag{61}$$

The linear problem (50) is of this form with $B_i\equiv 0$ $(i=1,2)$. Here, we choose $y_1=u^{(1)}$ and $y_2 = T_{rr}^{\mathrm{L}}(u^{(1)})$. This choice ensures that the y_i $(i=1,2)$ are continuous across an interface between dissimilar media, as they should be, even where the elastic constants λ, μ, A, B, and C themselves may be discontinuous; in this special case, which we explicitly consider below, (61) will not have any δ-function sources on its right-hand side.

Similarly, to solve the nonlinear problem (52) we choose $y_1=u^{(2)}$ and $y_2 = T_{rr}^{\mathrm{L}}(u^{(2)}) + T_{rr}^{\mathrm{NL}}(u^{(1)})$. Here, the physics of the situation requires that y_i $(i=1,2)$ be continuous and the mathematics of (61) guarantees that there are no δ-function sources here, either. Of course, $B_i\neq 0$ $(i=1,2)$ for this problem, as required by (52).

We use a commercial ODE integrator to solve these problems numerically. We limit our investigations to situations for which the elastic parameters tend to a constant value for large values of r. We simplify the calculations by requiring that λ, μ, A, B, and C be strictly constant for $r>b$ for some value of b chosen such that the elastic constants have assumed their asymptotic values. Mathematically this guarantees that

$$u^{(1)}(r)\equiv F/r,\quad r\geqslant b, \tag{62}$$

for some value of F. Computationally we take an arbitrary set of initial conditions, $y_i^{\alpha}(a)$ $(i=1,2)$ and we integrate forward into the region $r\geqslant b$. In general this solution has the property that

$$y_1^{\alpha}(r)\equiv G/r+Hr,\quad r\geqslant b, \tag{63}$$

where G and H are constants. We then take another, linearly independent set of initial conditions and repeat the process to get a second linearly independent solution $y_i^{\beta}(r)$. The desired solution to the linear problem, $[u^{(1)}(r),T_{rr}^{\mathrm{L}}(r)]$, is that linear combination of $y_i^{\alpha}(r)$ and $y_i^{\beta}(r)$ that satisfies (62). The proportionality between $y_1(a)$ and $y_2(a)$ for this solution yields μ^* via (54). [Alternatively we could have integrated backward from $r=b$ using the boundary condition (62) and the equivalent expression for $T_{rr}^{\mathrm{L}}(u^{(1)})$.]

This numerically determined $u^{(1)}(r)$ is used as the source for $u^{(2)}(r)$ in (52). Again, an arbitrary set of initial conditions $y_i(a)$ $(i=1,2)$ is used and (52) [put in the form (61) as discussed] is integrated forward to generate a particular solution to the inhomogeneous equation (52) which we denote $y_i^{\gamma}(r)$. The desired solution has the property

$$u^{(2)}(r)=\frac{(\lambda+3\mu+A+2B)}{2(\lambda+2\mu)}\frac{F^2}{r^3}+\frac{S}{r},\quad r\geqslant b, \tag{64}$$

for some value of S which is ultimately determined by the boundary conditions at $r=a$. The numerically generated solution $y_i^{\gamma}(r)$, however, has the property

$$y_1^{\gamma}(r)=\frac{(\lambda+3\mu+A+2B)}{2(\lambda+2\mu)}\frac{F^2}{r^3}+\frac{U}{r}+Vr,\quad r\geqslant b, \tag{65}$$

for some U and V. That is, the particular solution proportional to r^{-3} is the same in both (64) and (65) but the contributions from the homogeneous solutions are different in the two cases because they satisfy different boundary conditions. It is straightforward to add multiples of the homogeneous solutions, y_i^{α} and y_i^{β}, to the particular solution $y_i^{\gamma}(r)$ in order that the resultant solution satisfies both (64) and (53) [i.e., $y_2(a)=0$]. From (54) we can determine χ. This completes the description of how we numerically solve for the static deformation to second order in the pressure.

Let us now consider some specific solutions to cases for which the elastic moduli vary with position. We focus our attention on cases for which only the shear modulus varies, as this is the most important aspect of the problem for second-harmonic generation; λ, A, B, and C are constant. We consider two different functional forms for $\mu(r)$: exponential,

$$\mu(r)=\mu_\infty-\Delta\mu\,\exp[-(r-a)/l]; \tag{66}$$

and step-function,

$$\mu(r)=\begin{cases}\mu_\infty-\Delta\mu, & a<r<a+l,\\ \mu_\infty, & a+l<r<\infty.\end{cases} \tag{67}$$

We anticipate that the shear modulus near the borehole will generally be reduced relative to that far away: $\mu(a)=\mu_\infty-\Delta\mu<\mu_\infty$.

An advantage of considering (67) is that it can be solved analytically, simply by matching boundary conditions appropriately. Slightly more generally, suppose the elastic constants take on the values λ_1, μ_1, A_1, B_1, and C_1 in the inner layer $a<r<b$ $(b=a+l)$ and the values λ_2, μ_2, A_2, B_2, and C_2 in the outer layer $b<r$. The solution to the linear problem is

$$\begin{aligned}&u^{(1)}(r)=F/r+Gr,\quad a<r<b,\\ &u^{(1)}(r)=H/r,\quad b<r<\infty,\end{aligned} \tag{68}$$

where F, G, and H are determined from the continuity conditions of $u^{(1)}$ and $T_{rr}^{\mathrm{L}}(u^{(1)})$ at $r=b$, and the boundary condition for $T_{rr}^{\mathrm{L}}(u^{(1)})$ at $r=a$, given by (42). The solution to the second-order problem is

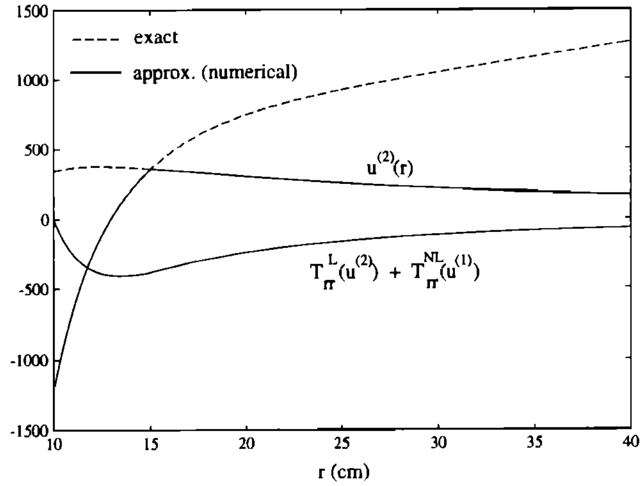

FIG. 3. The numerically generated second-order solution $u^{(2)}(r)$ for a particular choice of concentric-shell parameters, Eq. (67), corresponding to $a=10$ cm and $b=15$ cm. Both parts of the exact solution, Eq. (69), are shown over the entire range of r in order to emphasize the discontinuity of slope at $r=b$. Also shown is the numerically generated quantity $T_{rr}^{\mathrm{L}}(u^{(2)})+T_{rr}^{\mathrm{NL}}(u^{(1)})$. It is apparent that the boundary conditions at $r=a$, $r=b$, and $r\rightarrow\infty$ are satisfied.

$$u^{(2)}(r)=\frac{(\lambda_1+3\mu_1+A_1+2B_1)}{2(\lambda_1+2\mu_1)}\frac{F^2}{r^3}+\frac{F'}{r}+G'r,\quad a<r<b,$$
$$u^{(2)}(r)=\frac{(\lambda_2+3\mu_2+A_2+2B_2)}{2(\lambda_2+2\mu_2)}\frac{H^2}{r^3}+\frac{H'}{r},\quad b<r<\infty, \tag{69}$$

where F', G', and H' are determined from the continuity of $u^{(2)}$ at $r=b$, continuity of $[T_{rr}^{\mathrm{L}}(u^{(2)})+T_{rr}^{\mathrm{NL}}(u^{(1)})]$ at $r=b$, and the boundary condition for $[T_{rr}^{\mathrm{L}}(u^{(2)})+T_{rr}^{\mathrm{NL}}(u^{(1)})]$ at $r=a$, given by (53).

In Fig. 3 we plot $u^{(2)}(r)$ for a special case of (67). Both parts of the exact solution (69) are plotted as dashed lines over the entire interval of interest in order to emphasize the discontinuity of slope at $r=b$. The numerically generated solution is plotted as a solid line; its agreement with the exact solution lends confidence to our numerical procedure. It is worth noting that, because the shear modulus varies with position, not only is $u^{(2)}(a)$ not equal to zero but $u^{(2)}(r)$ takes on its largest value at $r=a$, for this example.

Winkler[30] has measured the dependence of the speeds of various sound waves in a sample of Berea Sandstone as a function of uniaxial stress, from which he was able to deduce the values of the second- and third-order elastic constants. His results are presented in Table II: they represent the only known values for a sedimentary rock. We shall base our calculations on these values. This sample is relatively weakly cemented, and its shear speed is less than that of water ($V_S=1.31$ km/s, $V_P=1.91$ km/s). Moreover, we note from (1) and (2) that the predicted pressure dependence of the speeds is quite large for this rock:

$$\rho_0\frac{dV_S^2}{dP}=1175,\quad \rho_0\frac{dV_P^2}{dP}=4381.$$

TABLE II. Deduced values of the second- and third-order elastic constants for Berea Sandstone.[a]

Elastic constant	Value (GPa)
μ_∞	3.66
λ	0.41
A	−4520
B	−1850
C	−1660

[a]Reference 30.

Indeed, it is much larger than that measured on the Boise Sandstone shown in Fig. 2, which in turn is much larger than the values shown in Table I for nongranular materials. We assume λ, A, B, and C are position independent and we take μ_∞ to be the value deduced by Winkler[30] from the (unstressed) shear speed. We defer our discussion of the linear response $u^{(1)}(a)$ until we develop a perturbation theory in the next subsection. We plot the second-order response at the borehole wall, χ, as a function of l for various values of $\Delta\mu/\mu_\infty$; the results for the exponential variation (66) are plotted in Fig. 4, those for the concentric shell, (67), in Fig. 5. The dashed lines are the results of a perturbation theory presented in the next subsection. The dotted line represents the quantity

$$\chi_0\equiv-\frac{\lambda+3\mu_\infty+A+2B}{8(\lambda+2\mu_\infty)}. \tag{70}$$

We wish to make two points: (a) If the depth of the altered zone, l, is either much smaller or much larger than the borehole radius a, then, in effect, the formation shear modulus is spatially uniform and $\chi\rightarrow0$, as seems intuitively obvious from the discussion in Sec. II A; (b) the overall scale for values of χ is set by the quantity χ_0, as seems obvious from Eq. (58), but it is leveraged by the degree and depth of al-

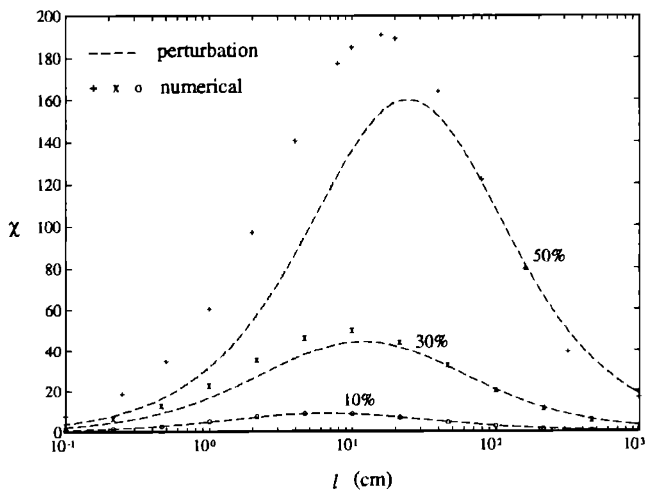

FIG. 4. The nonlinear expansion of the borehole wall, χ, defined by Eq. (54), as a function of l for the case of an exponential variation of the shear modulus with radius, Eq. (66). We consider the three cases $[\mu_\infty-\mu(a)]/\mu_\infty=10\%$, 30%, and 50%. The elastic parameters are taken from Table II and the borehole radius is $a=10$ cm. The dashed lines are the predictions of the perturbation theory, Eq. (99).

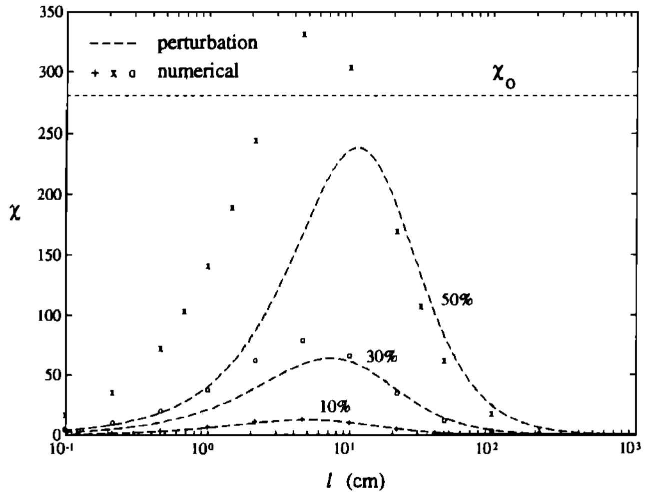

FIG. 5. Same as Fig. 4, but for the step variation, Eq. (67); l is the thickness of the annular region. The horizontal dashed line is the value χ_0, Eq. (70).

teration of the shear modulus. We make this connection more concrete in the next subsection.

D. Perturbation theory

Let us consider the case in which all the elastic parameters are nearly constant so that (55) and (57) nearly hold:

$$\begin{aligned} &\mu(\mathbf{r})=\bar{\mu}+\delta\mu(\mathbf{r}), \quad \lambda(\mathbf{r})=\bar{\lambda}+\delta\lambda(\mathbf{r}), \\ &A(\mathbf{r})=\bar{A}+\delta A(\mathbf{r}), \quad B(\mathbf{r})=\bar{B}+\delta B(\mathbf{r}), \\ &C(\mathbf{r})=\bar{C}+\delta C(\mathbf{r}). \end{aligned} \tag{71}$$

In this subsection we derive approximations to μ^* and χ that are accurate to first order in $\delta\mu(\mathbf{r})$. [We are anticipating the result (shown below) that only the cylindrically symmetric part of $\delta\mu$ contributes to the first-order perturbation theory.] We use a variational technique that obviates the necessity to calculate $\mathbf{u}^{(1)}(\mathbf{r})$ and $\mathbf{u}^{(2)}(\mathbf{r})$ to the desired accuracy.

The elastostatic energy is the integral of the energy density

$$W=\int U(\{C_l\};\{u_{i,j}\})dV, \tag{72}$$

where the set $\{C_l\}$ collectively represents the five elastic parameters of the problem. The expression for U valid to third order in strain for an isotropic material is given by Eq. (A3),

$$U=\frac{\lambda(\mathbf{r})}{2}I_1^2+\mu(\mathbf{r})I_2+\frac{A(\mathbf{r})}{3}I_3+B(\mathbf{r})I_1I_2+\frac{C(\mathbf{r})}{3}I_1^3 + O(E_{ij}^4), \tag{73}$$

where

$$\begin{aligned} &I_1=E_{ii}=\mathrm{tr}(\mathbf{E}), \quad I_2=E_{ij}E_{ji}=\mathrm{tr}(\mathbf{E}^2), \\ &I_3=E_{ij}E_{jk}E_{ki}=\mathrm{tr}(\mathbf{E}^3), \end{aligned} \tag{74}$$

in terms of the strain tensor (see Appendix), which we repeat here for convenience:

$$E_{ij}=\tfrac{1}{2}(u_{i,j}+u_{j,i}+u_{k,i}u_{k,j}). \tag{75}$$

The energy is also equal to the work done by the external forces in stressing the medium. For the case at hand, the borehole pressure is ramped up to some final value and the borehole wall moves out $[u_a\equiv u(a)]$ in a nonlinear fashion. The work per unit length of borehole is

$$W=\int_0^{u_a}P(u_a)2\pi a\,du_a. \tag{76}$$

We view (72) and (76) as microscopic and macroscopic expressions, respectively, of the same thing. Let us first evaluate each of these quantities for the simple case in which all the elastic parameters are independent of position. It shall prove extremely useful to express all quantities in terms of u_a. Thus (54) is

$$P(u_a)=\frac{2\mu^*}{a}u_a+\frac{8\chi(\mu^*)^3}{a^2\mu_\infty^2}u_a^2+\cdots. \tag{77}$$

Therefore (76) is simply

$$W=2\pi a\left(\frac{\mu^*}{a}u_a^2+\frac{8\chi(\mu^*)^3}{3a^2\mu_\infty^2}u_a^3+O(u_a^4)\right). \tag{78}$$

To evaluate the microscopic energy, (72), we need to express the displacement (38), (55), and (58) in terms of u_a:

$$u(r)=\frac{au_a}{r}+4\chi_0u_a^2\left(\frac{1}{r}-\frac{a^2}{r^3}\right)+O(u_a^3), \tag{79}$$

where

$$\chi_0\equiv-(\bar{\lambda}+3\bar{\mu}+\bar{A}+2\bar{B})/8(\bar{\lambda}+2\bar{\mu}), \tag{80}$$

by analogy with (70). In order to evaluate (73) let us rotate the strain tensor

$$\begin{aligned} &E_{rr}=\hat{\mathbf{r}}\cdot\mathbf{E}\cdot\hat{\mathbf{r}}=\frac{du}{dr}+\frac{1}{2}\left(\frac{du}{dr}\right)^2, \\ &E_{\theta\theta}=\hat{\theta}\cdot E\cdot\hat{\theta}=\frac{u}{r}+\frac{1}{2}\left(\frac{u}{r}\right)^2, \end{aligned} \tag{81}$$

as per (A12). The energy density is simply evaluated from (73) and (74),

$$U=\bar{\mu}(2a^2u_a^2r^{-4}+16\chi_0au_a^3r^{-4}-32\chi_0a^3u_a^3r^{-6}) + O(u_a^4), \tag{82}$$

as is the total energy,

$$W=\int_0^\infty U2\pi r\,dr=2\pi\bar{\mu}u_a^2+O(u_a^4). \tag{83}$$

Comparing (78) and (83) we see that

$$\mu^*\equiv\bar{\mu}, \tag{84}$$

and

$$\chi\equiv0, \tag{85}$$

as in Sec. II A. No surprises.

We now consider the primary problem of this subsection. All of the elastic parameters have a small radial variation, as in (71). We consider displacing the surface of the borehole by the *same* amount as before and at each point on the surface we consider by how much the applied stress changed to do this. For the case at hand, u_a is the same as before but P is different. The displacement has a change which is first order in $\delta\mu$, $\delta\lambda$, etc.,

$$u(r)\hat{\mathbf{r}} \to u(r)\hat{\mathbf{r}} + \delta\mathbf{u}(\mathbf{r}). \tag{86}$$

The change $\delta\mathbf{u}$ has components that are both first and second order in P. Because of the condition on the displacement, we have

$$\delta\mathbf{u}(r=a) \equiv \mathbf{0}. \tag{87}$$

We consider the changes in the macroscopic and in the microscopic expressions for the static energy. First, (78) holds for the macroscopic expression but

$$\mu^* = \bar{\mu} + \delta\bar{\mu}, \quad \chi \neq 0. \tag{88}$$

That is,

$$\Delta W = 2\pi a\left(\frac{\delta\bar{\mu}}{a}\, u_a^2 + \frac{8\chi(\mu^*)^3}{3a^2\mu_\infty^2}\, u_a^3 + O(u_a^4)\right). \tag{89}$$

[Note that $\chi \equiv \delta\chi$ because of (85).]

Our aim is to derive expressions for $\delta\bar{\mu}$ and χ exact to first order in $\delta\mu(\mathbf{r})$. To do so we need the change in the microscopic expression (72),

$$\Delta W = \int \frac{\partial U}{\partial C_l}\, \delta C_l\, dV + \int T_{ij}\delta u_{i,j}\, dV, \tag{90}$$

where

$$T_{ij} = \frac{\partial U}{\partial u_{i,j}}, \tag{91}$$

is the Piola–Kirchhoff stress tensor. Both T_{ij} and $\partial U/\partial C_l$ are to be evaluated using the constant values of the elastic parameters and the concomitant *unperturbed* solution [Eq. (79) in the case of the borehole problem]. Let us integrate the second term by parts:

$$\int T_{ij}\delta u_{i,j}\, dV = \int (T_{ij}\delta u_i)_{,j}\, dV - \int T_{ij,j}\delta u_i\, dV. \tag{92}$$

The first term can be converted into a surface integral that vanishes identically because $\delta\mathbf{u}(\mathbf{r})$ vanishes on the surface. The second term vanishes because the unperturbed solution satisfies (35). Therefore the change in the energy to first order in $\{\delta C_l\}$ is

$$\Delta W = \int \Delta U\, dV, \tag{93}$$

where

$$\Delta U = \frac{\delta\lambda(\mathbf{r})}{2}\, I_1^2 + \delta u(\mathbf{r}) I_2 + \frac{\delta A(\mathbf{r})}{3}\, I_3 + \delta B(\mathbf{r}) I_1 I_2 + \frac{\delta C(\mathbf{r})}{3}\, I_1^3 + O(E_{ij}^4), \tag{94}$$

where I_i $(i=1,2,3)$ are evaluated using the *unperturbed* solutions. For the case at hand only the second term in (94) contributes to $O(u_a^3)$ and we have

$$\Delta U = \delta\mu(\mathbf{r})(2a^2u_a^2r^{-4} + 16\chi_0 a u_a^3 r^{-4} - 32\chi_0 a^3 u_a^3 r^{-6}) + O(u_a^4). \tag{95}$$

At this point it is apparent that only the azimuthally and axially symmetric part of $\delta\mu$ contributes to the first-order perturbation result. So without loss of generality, we consider only a radially dependent $\delta\mu(r)$. Equation (95) implies

$$\Delta W = 4\pi a^2 u_a^2 \int_a^\infty \frac{\delta\mu(r)}{r^3}\, dr + 32\pi\chi_0 u_a^3 \int_a^\infty \delta\mu(r) \times \left(\frac{a}{r^3} - \frac{2a^3}{r^5}\right) dr. \tag{96}$$

Comparing (96) against (89) we have the desired result for the linear response:

$$\delta\bar{\mu} = 2a^2 \int_a^\infty \frac{\delta\mu(r)}{r^3}\, dr, \tag{97}$$

which can be rewritten

$$\mu^* = 2a^2 \int_a^\infty \frac{\mu(r)}{r^3}\, dr. \tag{98}$$

Note that this result is independent of the choice of $\bar{\mu}$. Similarly the quadratic response can be expressed in this perturbation theory as

$$\chi = \frac{6\mu_\infty^2 a}{(\mu^*)^3}\, \chi_0 \int_a^\infty \delta\mu(r)\left(\frac{a}{r^3} - \frac{2a^3}{r^5}\right) dr. \tag{99}$$

This result is also independent of the choice of $\bar{\mu}$.

Let us compare the predictions of these perturbation theory results against our numerically calculated results. First, for the linear response (98) it is convenient to collapse the data by defining

$$x = \frac{2a^2}{[\mu_\infty - \mu(a)]} \int_a^\infty \frac{\mu_\infty - \mu(r)}{r^3}\, dr. \tag{100}$$

We have $x \approx 0$ for shallow depth of alteration and $x \approx 1$ for deep. Equation (98) reads

$$[\mu^* - \mu(a)]/[\mu_\infty - \mu(a)] = 1 - x. \tag{101}$$

With this definition the constant background $\bar{\mu}$ has been removed. The quantity x can be evaluated in closed form for the concentric shell problem (67) and it can be expressed in terms of an exponential integral for the exponential variation, (66). In Fig. 6 we plot all our data for the linear response μ^* and we compare against (101). Even for the case of large values of $\Delta\mu/\mu_\infty$, (101) works quite well.

Baker[31] has considered the effect of an invaded zone on linear borehole acoustic waveforms. Our result in (98) is relevant to that part of his waveforms dominated by the low-frequency tube wave.

For the quadratic response of the system, the integrals in (99) can be done in closed form for the step variation (67) and can be expressed in terms of exponential integrals for the exponential variation (66). These perturbation theory results are also plotted in Figs. 4 and 5.

We have several observations.

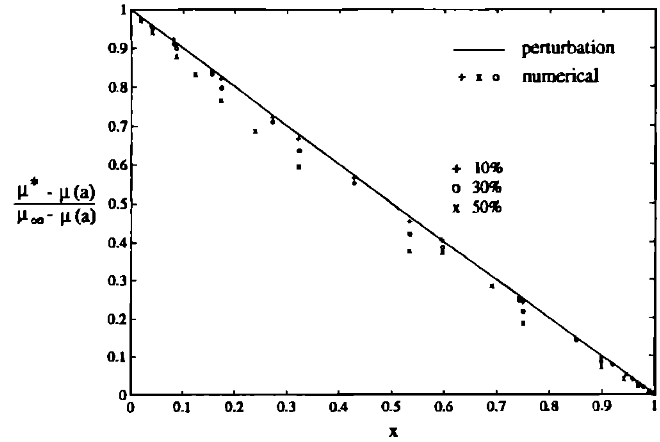

FIG. 6. Results of the calculated linear response μ^* plotted as $[\mu^*-\mu(a)]/[\mu_\infty-\mu(a)]$ vs x for the case of exponential variation of the shear modulus, Eq. (66), and step variation, Eq. (67). The straight line is the perturbation theory result, Eq. (98). The degree of alteration is 10%, 30%, and 50% as in Figs. 4 and 5.

(a) The perturbation theories work well as long as $\delta\mu(r)$ is small, regardless of the depth of the alteration but they do *not* apply if the depth is small but the degree of alteration $\Delta\mu/\mu_\infty$ is large. This is apparent from Figs. 4 and 5 and especially 6 where we can see that the slopes near $x=0$ and $x=1$ are not correctly given by (101) when the degree of alteration is large.

(b) Even when the perturbation theory is inapplicable, it gives quite reasonable estimates of the required quantities.

(c) Equation (99) makes clear our earlier claim that the overall scale for χ is set by the quantity χ_0, leveraged by the degree and depth of alteration of the shear modulus as expressed by the integral in (99).

(d) There is no first-order contribution to χ from $\delta\lambda(r)$, $\delta A(r)$, $\delta B(r)$, or $\delta C(r)$, as was already obvious from (60).

(e) If there is an axial or azimuthal variation in the elastic parameters, only the radially dependent part contributes, to first order in the variation.

E. Significance of $A+2B$

Evidently the important combination of nonlinear parameters would seem to be $A+2B$, as is suggested by the perturbation theory, (99). We conclude this section with a description of a simple measurement that would yield this combination directly. We remind the reader that the third-order elastic constants allow one to compute the changes in the speeds of sound to first order in applied stress.[11,12] Equations (1) and (2) are specific examples wherein the applied stress is an isotropic pressure but similar results have been derived for the case of an applied uniaxial stress. In the case at hand, the application of a quasistatic pressure in the borehole causes the formation to expand radially. If the shear modulus is spatially uniform then the (linear) deformation is purely a shear, i.e., $\nabla\cdot\mathbf{u}=0$, as is obvious since (55) does not involve λ. The deformation is such that a thin arc segment gets thinner in the radial direction and longer in the circumferential direction just such that the volume remains unchanged. Meanwhile the displacement is itself in the thinning direction.

We are led to consider the following case of plane wave propagation in an unbounded medium: We apply a static shear deformation in the x_1-x_2 plane of the form $\mathbf{u}_0=(sx_1,-sx_2,0)$ and we calculate the change in the speed of small-amplitude longitudinal sound propagating in the x_1 direction. Each point at position $\mathbf{r}$ has moved to a position $\mathbf{r}''=\mathbf{r}'+(\tilde{u}(x_1',t),0,0)$, where $\mathbf{r}'=[(1+s)x_1,(1-s)x_2,x_3]$ is the new coordinate system, and $\tilde{u}(x_1',t)$ is the small-amplitude wave in the new coordinate system. Following exactly the derivation of Hughes and Kelly[11] we find that the change of the speed of longitudinal sound in the new coordinate system is

$$\rho_0 \left.\frac{dV_P^2}{ds}\right|_{s=0}=2(2\lambda+5\mu+A+2B). \tag{102}$$

We may rewrite this as

$$\left.\frac{1}{V_P}\frac{dV_P}{ds}\right|_{s=0}=1-8\chi_0. \tag{103}$$

We see from (102) and (103) that the combination of parameters $A+2B$ has a simple physically intuitive interpretation. We note that $A+2B=2C_{155}$ in Brugger's[32] notation, and $A+2B=2m$ in Murnaghan's[33] notation. See Green[34] for a useful table that allows a comparison of the different conventions for the third-order elastic constants.

Let us consider the difference between a dry and a liquid saturated porous medium. The equation that led to (102) is of the form

$$\rho_0 \frac{\partial^2 u}{\partial t^2}=K(s)\frac{\partial^2 u}{\partial x^2}, \tag{104}$$

where $x=x_1'$ and $K(s)$ is an effective P-wave stiffness in the new coordinate system: $V_P^2(s)=K(s)/\rho_0$. When a shear stress of the form described above is applied to a sample there is *no* change in the volume to the required order of accuracy for (102). Therefore, there is essentially no change in the contribution to the P-wave stiffness from the fluid because the pore pressure has not changed because of the static shear. This would seem to imply that dK/ds is the same for a wet or dry porous medium. It is known[35] that fluid-saturated rocks have very much smaller pressure dependences than dry rocks, basically because an externally applied pressure is partially supported by the pore fluid. Therefore the individual third-order constants A, B, and C can be expected to be very different, as implied by (1) and (2). In the case of sedimentary rocks in which the dry frame has enormous values of A and B as in Table II, we believe that the combination $A+2B$, however, is essentially independent of fluid saturation.

III. DISCUSSION

In this section we connect the results of Secs. I and II and we estimate the rate with which tube wave speed changes with pressure and also the efficiency with which second-harmonic generation may occur. First we emphasize that A_f and B_f of Sec. I are *not* to be confused with A, B, or C of Sec. II. It is possible, though unconventional, to describe second-order fluid elasticity by means of the Lagrangian formulation of Sec. II. Kostek *et al.*[36] have shown that for a fluid the Lagrangian elastic parameters are related to the two Eulerian ones by

$$\lambda=A_f,\quad \mu=0,$$
$$A=0,\quad B=-A_f,\quad C=(A_f-B_f)/2. \tag{105}$$

Second, the response of the solid is more conveniently expressed in terms of the Lagrangian pressure P, as per (54), whilst the tube wave properties are expressed in terms of the Eulerian pressure p, as per (9). Using Eqs. (9), (34), and (54) we have

$$N_1=1/\mu^* \tag{106}$$

and

$$N_2=1/(2\mu^*)-2\mu^*\chi/\mu_\infty^2. \tag{107}$$

Therefore (24) may be rewritten as

$$\alpha=\rho_0\left[\frac{1}{A_f^2}\left(2+\frac{B_f}{A_f}\right)+\frac{3}{\mu^*A_f}+\left(\frac{3}{2(\mu^*)^2}+\frac{4\chi}{\mu_\infty^2}\right)\right]. \tag{108}$$

Let us estimate the value of this parameter with the available information. We consider a water-filled borehole in a rock whose properties are the same as those of Table II. The parameters for water[14] are A_f=2.27 GPa (1 GPa=10^{10} dyn/cm^2) and B_f/A_f=5.2. With these values we have the fluid contribution to (108):

$$\alpha_f\equiv\frac{\rho_0}{A_f^2}\left(2+\frac{B_f}{A_f}\right)=1.4\times10^{-15}\ \text{s}^2/\text{N}. \tag{109}$$

If the shear modulus has no radial variation then $\mu^*\equiv\mu_\infty$ and $\chi\equiv0$, and there is only a small change in the nonlinear parameter α. We find

$$\alpha[\mu(r)\equiv\mu_\infty]=1.86\times10^{-15}\ \text{s}^2/\text{N}, \tag{110}$$

there being a small nonlinearity due to the shear modulus of the solid. If, however, there is a modest amount of alteration in the region near the borehole then, as we have seen, the value of χ can be quite large and it will dominate the nonlinear response. Using a ray-tracing approach Hornby[37] has found that the compressional speed of sound in the vicinity of the borehole can be very significantly reduced, relative to that farther away, due presumably to a combination of stress relief, drilling induced damage, and (in the case of some shales) swelling. More than likely there is at least as great an alteration of the shear modulus as there is the compressional.

To be specific, we consider that the shear modulus has been reduced by 30% relative to that of the unaltered formation. We consider the exponential variation for $\mu(r)$, Eq. (66). For the case in which the decay length l equals a, our calculations give μ^*=3.0 GPa and χ=50. In this case the nonlinearity is dominated by that of the solid

$$\alpha(\text{altered zone})=16.9\times10^{-15}\ \text{s}^2/\text{N}. \tag{111}$$

Thus, assuming the approximate validity of the values of the parameter set with which we have been working, the value of the nonlinearity parameter can be much enhanced relative to that of the fluid alone. We caution that the rock in question is rather poorly consolidated and is air-saturated. A water-saturated sample may well have much reduced values of the parameters A, B, and C, although we have argued that the combination $A+2B$ should be the same.

Let us estimate the relative amplitude of a second-harmonic tube wave relative to that of the fundamental, from Eq. (32). We take P_0=100 Pa and $\omega=2\pi$ kHz. In the near field $z\approx0$ one has

$$|P(2\omega)/P(\omega)|=2.7\times10^{-7}, \tag{112}$$

whereas farther along the borehole, $z=10^3$ cm, say,

$$|P(2\omega)/P(\omega)|=3.0\times10^{-5}. \tag{113}$$

These values are well within the ability to measure with lock-in techniques. Naturally, the amplitude of second-harmonic generation is enhanced for higher frequencies; most laboratory measurements are done in the megahertz frequency range. In this paper, however, the low-frequency assumptions limit the validity of our results to the kilohertz range, for a 10-cm-radius borehole.

Finally, we consider the pressure dependence of the tube wave speeds. Equation (16) may be rewritten as

$$\rho_0\left.\frac{dV_T^2}{dp_0}\right|_{p_0=0}=\frac{B_f/A_f-A_f/\mu^*-A_f^2/2(\mu^*)^2+4(A_f/\mu^*)^2\chi}{(1+A_f/\mu^*)^2}. \tag{114}$$

As we mentioned earlier the value for a plane wave in water is simply

$$\rho_0\frac{dc^2}{dp}\equiv\frac{B_f}{A_f}=5.2, \tag{115}$$

which value would apply for a rigid borehole. In the case that the formation shear modulus is radially independent with a value given by μ_∞ of Table II, we find

$$\rho_0\frac{dV_T^2}{dp}=1.66, \tag{116}$$

which is a rather significant reduction relative to (115). In fact, in this case Eq. (114) can be alternatively written as

$$\rho_0\left.\frac{dV_T^2}{dp_0}\right|_{p_0=0}=\left(\frac{B_f}{A_f}+\frac{1}{2}\right)\left(\frac{V_T}{c}\right)^4-\frac{1}{2}\quad(\chi\equiv0). \tag{117}$$

If, however, we consider the values appropriate to the altered zone discussed above we find

$$\rho_0\frac{dV_T^2}{dp}=25.1. \tag{118}$$

We see from (117) that an unaltered zone will always act to *reduce* the pressure dependence of the speed relative to that of the fluid alone and that modest alteration can enhance it enormously. In principle, either measurement, that of pressure-dependent speed or that of second-harmonic generation, will yield the same information about the formation's nonlinear parameter, namely, χ.

ACKNOWLEDGMENTS

We are grateful for stimulating discussions with B. I. Halperin, who led us to understand the limitations of the perturbation theory, and with B. K. Sinha. This work was performed while one of us (A.N.N.) was a visiting scientist at Schlumberger–Doll Research.

APPENDIX: SECOND-ORDER EQUATIONS OF ELASTICITY IN CYLINDRICAL COORDINATES

In this Appendix we develop the equations of motion for nonlinear elasticity appropriate to cylindrical symmetry. Our results are accurate through second order in the displacement field. First, we present some general results, then we review the equations in cartesian coordinates, finally we present results for the cylindrically symmetric case.

Let $\mathscr{P}$ be a typical particle of an elastic body and let $\mathbf{X}$ and $\mathbf{x}$ be its material (Lagrangian) and current (Eulerian) position vectors. The motion is thus defined by $x=x(\mathbf{X},t)$, and the relative displacement vector is $\mathbf{u}=\mathbf{x}-\mathbf{X}$. The deformation gradient $\mathbf{F}$ is defined by

$$\mathbf{F}=\frac{\partial\mathbf{x}}{\partial\mathbf{X}}=\mathbf{I}+\frac{\partial\mathbf{u}}{\partial\mathbf{X}}, \tag{A1}$$

where $\mathbf{I}$ is the identity tensor. In component form (A1) is given by $F_{ij}=\delta_{ij}+u_{i,j}$, where $_{,j}$ denotes $\partial/\partial X_j$. The relative Lagrangian (Green's) strain tensor is defined as

$$\mathbf{E}=\tfrac{1}{2}(\mathbf{F}^T\mathbf{F}-\mathbf{I}). \tag{A2}$$

It measures local distortions in the neighborhood of a material point through changes in differential intervals $d\mathbf{x}^2-d\mathbf{X}^2=2E_{kl}\,dX_k\,dX_l$.

For a hyperelastic material, the strain energy function U, defined per unit volume of the undeformed material, is a function of the Green's strain tensor only, $U=U(\mathbf{E})$. For an isotropic material it can only depend on the rotational invariants of that tensor. We consider the Taylor's series expansion for an isotropic material in the absence of any prestress as given by Landau and Lifshitz[10]

$$U=\frac{\lambda}{2}I_1^2+\mu I_2+\frac{A}{3}I_3+BI_1I_2+\frac{C}{3}I_1^3+O(E_{ij}^4), \tag{A3}$$

where

$$I_1=E_{ii},\quad I_2=E_{ij}E_{ij},\quad I_3=E_{ij}E_{jk}E_{ki}. \tag{A4}$$

Here, λ and μ are the usual second-order Lamé constants and A, B, and C are the third-order elastic constants. All quantities of interest, the stress-strain relation and the equation of motion are derivable from the energy density. Specifically, the nonsymmetric first Piola–Kirchhoff (pseudo-) stress tensor T is given by[38]

$$T_{ij}=\frac{\partial U}{\partial u_{i,j}}, \tag{A5}$$

which can be rewritten in a form more useful for our purposes as (Kelvin–Cosserat)[38]

$$T_{ij}=\frac{\partial U}{\partial E_{jk}}F_{ik}. \tag{A6}$$

Finally, the Lagrangian form of the equation of motion is

$$\rho_0\mathbf{u}_{,tt}=\nabla\cdot\mathbf{T}+\rho_0\mathbf{f}, \tag{A7}$$

where $\mathbf{f}$ is a body force per unit mass of undeformed material. Equations (A1)–(A7) together with proper initial and boundary conditions form the basis for studying nonlinear dynamic phenomena in solids. These foregoing points are discussed in more detail by Eringen and Suhubi.[38]

The second-order elasticity theory is obtained by neglecting all cubic and higher-order terms on the right-hand side of equations (A6) and (A7). For completeness we will first present them in Cartesian coordinates. From (A1) and (A2) we obtain the following expression for the components of the Green's strain tensor

$$E_{ij}=\tfrac{1}{2}(u_{i,j}+u_{j,i}+u_{k,i}u_{k,j}). \tag{A8}$$

Equations (A3)–(A6) give

$$\begin{aligned}T_{ij}={}&\lambda u_{k,k}\delta_{ij}+\mu(u_{i,j}+u_{j,i})+\left(\frac{\lambda}{2}u_{k,l}u_{k,l}+Cu_{k,k}u_{l,l}\right)\\&\times\delta_{ij}+Bu_{k,k}u_{j,i}+\frac{A}{4}u_{j,k}u_{k,i}+\frac{B}{2}(u_{k,l}u_{k,l}\\&+u_{k,l}u_{l,k})\delta_{ij}+(\lambda+B)u_{k,k}u_{i,j}\\&+\left(\mu+\frac{A}{4}\right)(u_{i,k}u_{j,k}+u_{k,i}u_{k,j}+u_{i,k}u_{k,j}),\end{aligned} \tag{A9}$$

where only terms quadratic in the displacement gradient were kept. The equations of motion in terms of the displacements are obtained by substituting (A9) into (A7):

$$\begin{aligned}&u_{i,tt}-(V_P^2-V_S^2)u_{k,ki}-V_S^2u_{i,kk}-f_i\\&=(1/\rho_0)[(\lambda+\mu+B+A/4)(u_{l,k}u_{l,ik}+u_{i,k}u_{l,lk})\\&\quad+(\lambda+B)u_{l,l}u_{i,kk}+(B+2C)u_{l,l}u_{k,ki}+\tfrac{1}{4}(A+4B)\\&\quad\times(u_{k,l}u_{l,ik}+u_{l,i}u_{k,kl})+(\mu+A/4)(u_{l,i}u_{l,kk}\\&\quad+2u_{k,l}u_{i,kl}+u_{i,l}u_{l,kk})],\end{aligned} \tag{A10}$$

where $V_P^2=(\lambda+2\mu)/\rho_0$ and $V_S^2=\mu/\rho_0$. Equation (A10) is derived in Goldberg[39] and is used by Jones and Kobett,[4] among others.

We will now present the equations in terms of a cylindrical coordinate system. The position of a material particle in its original configuration is expressed by $\mathbf{X}=(r,\theta,z)$. The position in its current configuration is $\mathbf{x}=\mathbf{X}+\mathbf{u}$ in which the displacement coordinates are referenced to the original coordinates: $\mathbf{u}=(u_r,u_\theta,u_z)$. In this appendix we consider only the axisymmetric case such that all the dynamical quantities depend only upon r and z and the displacement $\mathbf{u}$ has components only in the r and z directions. First, we compute the components of the tensor $\mathbf{F}$ in cylindrical coordinates using $F_{rz}=\hat{\mathbf{r}}\cdot\mathbf{F}\cdot\hat{\mathbf{z}}$, etc. The only nonzero components of this tensor are

$$\begin{aligned}&F_{rr}=1+u_{r,r},\quad F_{rz}=u_{r,z},\quad F_{\theta\theta}=1+u_r/r,\\&F_{zr}=u_{z,r},\quad F_{zz}=1+u_{z,z}.\end{aligned} \tag{A11}$$

The components of the strain tensor are given by (A2):

$$E_{rr}=u_{r,r}+\tfrac{1}{2}(u_{r,r}^2+u_{z,r}^2),$$

$$E_{rz}=\tfrac{1}{2}(u_{r,z}+u_{z,r}+u_{r,r}u_{r,z}+u_{z,r}u_{z,z}),$$

$$E_{\theta\theta}=u_r/r+\tfrac{1}{2}(u_r/r)^2,\quad E_{zr}=E_{rz},\tag{A12}$$

$$E_{zz}=u_{z,z}+\tfrac{1}{2}(u_{r,z}^2+u_{z,z}^2).$$

The strain energy function can be expressed in terms of displacement gradients by means of Eqs. (A3), (A4), and (A12); the relationships between stresses and displacements are obtained from (A6) and (A11). For example, the component T_{rz} is obtained as

$$T_{rz}=\frac{\partial U}{\partial E_{zr}}F_{rr}+\frac{\partial U}{\partial E_{zz}}F_{rz}.\tag{A13}$$

Neglecting all cubic and higher-order terms in the displacement gradient we find

$$\begin{aligned}T_{rr}={}&\lambda(u_{r,r}+u_r/r+u_{z,z})+2\mu u_{r,r}+(\lambda/2)\\&\times[u_{r,r}^2+(u_r/r)^2+u_{r,z}^2+u_{z,r}^2+u_{z,z}^2]+C(u_{r,r}+u_r/r\\&+u_{z,z})^2+(A/4)(u_{r,r}^2+u_{r,z}u_{z,r})+B[u_{r,r}^2+(u_r/r)^2\\&+\tfrac{1}{2}(u_{r,z}+u_{z,r})^2+u_{z,z}^2]+(\lambda+2B)(u_{r,r}+u_r/r\\&+u_{z,z})u_{r,r}+(\mu+A/4)(3u_{r,r}^2+u_{r,z}^2+u_{z,r}^2\\&+u_{r,z}u_{z,r}),\end{aligned}\tag{A14}$$

$$\begin{aligned}T_{rz}={}&\mu(u_{r,z}+u_{z,r})+(A/4)(u_{r,r}+u_{z,z})u_{z,r}+(\lambda+B)\\&\times(u_{r,r}+u_r/r+u_{z,z})u_{r,z}+B(u_{r,r}+u_r/r+u_{z,z})u_{z,r}\\&+(\mu+A/4)[2u_{r,r}u_{r,z}+2u_{z,z}u_{r,z}\\&+u_{z,r}(u_{r,r}+u_{z,z})],\end{aligned}\tag{A15}$$

$$\begin{aligned}T_{\theta\theta}={}&\lambda(u_{r,r}+u_r/r+u_{z,z})+2\mu u_r/r+(\lambda/2)\\&\times[u_{r,r}^2+(u_r/r)^2+u_{r,z}^2+u_{z,r}^2+u_{z,z}^2]+C(u_{r,r}+u_r/r\\&+u_{z,z})^2+(A/4)(u_r/r)^2+B[u_{r,r}^2+(u_r/r)^2+\tfrac{1}{2}(u_{r,z}\\&+u_{z,r})^2+u_{z,z}^2]+(\lambda+2B)(u_{r,r}+u_r/r+u_{z,z})u_r/r\\&+3(\mu+A/4)(u_r/r)^2,\end{aligned}\tag{A16}$$

$$\begin{aligned}T_{zr}={}&\mu(u_{r,z}+u_{z,r})+(A/4)(u_{r,r}+u_{z,z})u_{r,z}+(\lambda+B)\\&\times(u_{r,r}+u_r/r+u_{z,z})u_{z,r}+B(u_{r,r}+u_r/r+u_{z,z})u_{r,z}\\&+(\mu+A/4)[2u_{r,r}u_{z,r}+2u_{z,z}u_{z,r}\\&+u_{r,z}(u_{r,r}+u_{z,z})],\end{aligned}\tag{A17}$$

$$\begin{aligned}T_{zz}={}&\lambda(u_{r,r}+u_r/r+u_{z,z})+2\mu u_{z,z}+(\lambda/2)\\&\times[u_{r,r}^2+(u_r/r)^2+u_{r,z}^2+u_{z,r}^2+u_{z,z}^2]+C(u_{r,r}+u_r/r\\&+u_{z,z})^2+(A/4)(u_{r,z}u_{z,r}+u_{z,z}^2)+B[u_{r,r}^2+(u_r/r)^2\\&+\tfrac{1}{2}(u_{r,z}+u_{z,r})^2+u_{z,z}^2]+(\lambda+2B)\\&\times(u_{r,r}+u_r/r+u_{z,z})u_{z,z}+(\mu+A/4)\\&\times(u_{r,z}^2+u_{r,z}u_{z,r}+u_{z,r}^2+3u_{z,z}^2).\end{aligned}\tag{A18}$$

These equations were also obtained independently, starting from the Cartesian ones and carrying out the proper coordinate transformations.

Finally the equations of motion are given by considering the net force on a material element whose undistorted volume is confined to $[r,r+dr]$, $[z,z+dz]$, and $[\theta,\theta+d\theta]$. The Piola–Kirchhoff stress is defined such that $\mathbf{T}\cdot\hat{\mathbf{n}}\,dA$ gives the net force on an element whose undistorted area is dA and whose undistorted outward normal is $\hat{\mathbf{n}}$. In terms of the components of $\mathbf{T}$ we have

$$\rho_0 u_{r,tt}=T_{rr,r}+T_{rz,z}+(1/r)(T_{rr}-T_{\theta\theta})+\rho_0 f_r,\tag{A19}$$

$$\rho_0 u_{z,tt}=T_{zr,r}+T_{zz,z}+(1/r)T_{zr}+\rho_0 f_z.\tag{A20}$$

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