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Shear wave propagation in a periodically layered medium – an asymptotic theory

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The propagation of low frequency or long wavelength disturbances in periodically layered media is considered. An asymptotic series is derived for the frequency of the first branch of the Bloch wave spectrum. The expansion is in dimensionless wavenumber and is developed explicitly for SH waves traveling obliquely through layerings with arbitrary periodic stratification. The first dispersive term is discussed in detail for a two phase medium and numerical results are presented showing that the asymptotic approximation to the dispersion equation can accurately approximate pulse dispersion. The asymptotic theory presented here may be used to define a dynamic effective medium, as opposed to an equivalent static effective medium.

1. Introduction

Wave propagation in periodic media is, in some senses, a rather heuristic process, particularly so at low frequencies where the concept of an homogenized effective medium is natural. In this regime the wavelength is "large" in comparison with the unit cell size, and the wave "sees" a smoothed medium with density equal to the spatial average and elastic constants determined from purely static considerations, see e.g. [1]. Size effects become important at higher values of frequency where wave interference effects dominate, resulting in the purely dynamic phenomenon of band gaps and passing bands [2]. The range of wave phenomena encountered in periodic materials are all contained in the Bloch wave description of the medium [2, 3]. The Bloch waves are defined for a given medium as the solutions to specific eigenvalue problems on the unit cell. Each of the denumerable set of Bloch waves has its own associated dispersion curve, and these curves in themselves provide a wealth of information on the wave propagation characteristics of the medium. Obtaining the dispersion curves is often a non-trivial matter, e.g. [4, 5], and in the process one can tend to lose track of the physical nature of the mode. For instance, the "effective medium" behavior at low frequency is very difficult to discern from a transcendental equation for frequency as an implicit function of wavenumber. Considering that the effective medium is related to the static response of the medium, it is reasonable to expect that the first corrections to the low frequency dispersion can be described by a theory which is closer to static than fully dynamic. It is also reasonable to expect that this description provides a simpler way of looking at low frequency wave propagation in periodic media.

In this paper we develop an asymptotic theory to determine the first purely dispersive effects above and beyond the effective medium prediction, which is non-dispersive. The method is presented for the particular

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case of anti-plane shear wave propagation in a periodically layered medium. The unit period may be composed of a discrete set of layers, each with constant properties, or by a continuous variation in material parameters. A major result is an explicit formula for the first nonlinear term in the dispersion equation for obliquely traveling waves in arbitrary periodic layers, generalizing the earlier result of Santosa and Symes [6] for propagation normal to the layering. This level of approximation is adequate to accurately describe the evolution of long wavelength initial disturbances through many layers [6], and numerical comparisons with "exact" results are presented here which demonstrate the utility and simplicity of the asymptotic approximation. This paper considers in great detail the specific case of SH-polarized wave propagation, which is the simplest example of elastic wave propagation possible. The case of fully elastic waves in anisotropic layered media exhibits essentially the same dispersive phenomena as those illustrated in detail in this paper, but the simplicity is lost in the algebraic complexity. For further details, we refer the reader to [7, 8].

The asymptotic form of the dispersion equation can be thought of as the dispersion equation for an effectively homogeneous and anisotropic medium. It gives the phase speed for wave propagation in a given direction at a given frequency. Alternatively, it provides the vertical wavenumber for waves of a given horizontal slowness. The latter interpretation is of practical use in numerical simulation of waves of fixed horizontal slowness, or "pseudo plane waves", introduced in Section 4. The similarities and differences for plane wave and pseudo plane wave behavior are discussed in some detail. In particular, we deduce the conditions under which the dispersion vanishes for each type of wave in a two phase composite.

The layout of the paper is as follows. The problem and the appropriate scalings are defined in Section 2. The asymptotic solution to the periodic eigenvalue problem is described in Section 3, and the analytic nature of the dispersion equation is discussed in Section 4. Applications to discretely layered media are also discussed in Section 4 for both plane waves and pseudo plane waves. Finally, numerical results are presented in Section 5 for waves travelling through many layers in two materials.

2. The asymptotic theory

We consider the problem of wave propagation in a periodically layered elastic medium. The medium is assumed to be layered in the x_2 direction, the layers are of thickness h, with all material parameters independent of x_1 and x_3. The relevant parameters are the density $\rho(x_2)$ and the shear modulus $\mu(x_2)$, which we build out of periodic functions of unit period,

$$\rho(x+1)=\rho(x), \qquad \mu(x+1)=\mu(x).$$

Let the SH (horizontally polarized shear) displacement be $u(x_1, x_2, t)$ where, with no loss in generality, we are assuming the out-of-plane polarization is always in the x_3 direction. The only non-zero components of the stress tensor are

$$\sigma_{13}=\sigma_{31}=\mu\left(\frac{x_2}{h}\right)\frac{\partial u}{\partial x_1}, \qquad \sigma_{23}=\sigma_{32}=\mu\left(\frac{x_2}{h}\right)\frac{\partial u}{\partial x_2}, \tag{1}$$

and the only nontrivial equation of motion is

$$\frac{\partial \sigma_{13}}{\partial x_1}+\frac{\partial \sigma_{23}}{\partial x_2}=\rho\left(\frac{x_2}{h}\right)\frac{\partial^2 u}{\partial t^2}. \tag{2}$$

Elimination of the stresses in (1) and (2) leads to the partial differential equation for $u(x_1, x_2, t)$

$$\mu\left(\frac{x_2}{h}\right)\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial}{\partial x_2}\left(\mu\left(\frac{x_2}{h}\right)\frac{\partial u}{\partial x_2}\right)=\rho\left(\frac{x_2}{h}\right)\frac{\partial^2 u}{\partial t^2}. \tag{3}$$

In a recent paper [6], Santosa and Symes constructed an asymptotic theory for initial value problems for hyperbolic partial differential equations with periodic coefficients. They showed that an accurate description of the dispersion effects can be modeled by a *dynamic* effective medium equation. Their theory, which is based on the Bloch expansion, is summarized in the context of our problem.

Assume that the initial data $u(x_1, x_2, 0)$ and $\partial u/\partial t(x_1, x_2, 0)$ are band-limited; i.e., their Fourier transforms are of compact support. Denote by λ the wavelength of the Fourier component corresponding to the highest wavenumber. The main assumption in [6] is that

$$\varepsilon := \frac{h}{\lambda} \ll 1.$$

When this holds the O(1) solution of the initial value problem, which is valid to $t = O(\varepsilon^{-2})$, can be written down explicitly. By O(1) solution we mean that the correction to this approximate solution is formally $O(\varepsilon)$.

In order to write down the leading order asymptotic solution we must first consider the eigenvalue problem for the Bloch waves $w(y_1, y_2)$,

$$\mu(y_2)\frac{\partial^2 w}{\partial y_1^2}+\frac{\partial}{\partial y_2}\left(\mu(y_2)\frac{\partial w}{\partial y_2}\right)=-\rho(y_2)\omega^2(k_1,k_2)w. \tag{4a}$$

with the "monodromy" relations

$$w(y_1+1, y_2)=w(y_1, y_2)\exp(\mathrm{i}k_1),\quad w(y_1, y_2+1)=w(y_1, y_2)\exp(\mathrm{i}k_2). \tag{4b}$$

The system (4) admits an infinity of solutions for each wavenumber pair (k_1, k_2). The eigen-pair $(w(y_1, y_2), \omega(k_1, k_2))$ corresponding to the smallest $\omega(k_1, k_2)$ is called the *first branch*. A main result in [6] is that an approximate solution of the initial value problem which takes into account only the first branch Bloch waves is correct to O(1) *uniformly* in t. This leads to a tremendous amount of simplifications.

For instance, let the first branch of the dispersion relation be the frequency $\omega(k_1, k_2)$. The leading order solution to an initial value problem for which the initial velocity is zero is given by

$$u(x_1, x_2, t)=\frac{1}{4\pi^2}\int F(k_1, k_2)\exp\left(\mathrm{i}k_1x_1+\mathrm{i}k_2x_2\pm\mathrm{i}\frac{1}{h}\,\omega(hk_1, hk_2)t\right)\mathrm{d}k_1\,\mathrm{d}k_2. \tag{5}$$

Here $F(k_1, k_2)$ is the Fourier transform of the initial data $u(x_1, x_2, 0)$, and the $\pm$ sign denotes two components of the solution, which are to be summed.

Because we are only interested in the leading order solution, it turns out that we only need to find the power series expansion of $\omega(k_1, k_2)$ about $k_1 = k_2 = 0$ to three terms. Inserting the expansion in (5) then provides us with an explicit representation for the asymptotic solution to the initial value problem. The next three sections deal with the question of determining the power series expansion for ω and its properties.

3. The asymptotic expansion

In this section we employ a Neumann expansion to solve for $\omega(k_1, k_2)$. The method is the same as that of Norris [7] who considered the general problem of multimode wave propagation in periodically layered anisotropic solids. We first recast (4) as a first order system. We assume $w(y_1, y_2)$ to be of the form

$$w(y_1, y_2) = (ik)^{-1} v(y_2)\, e^{ik_1 y_1},$$

and use a second variable $\tau(y_2)$ defined by

$$\mu(y_2) \frac{\partial w}{\partial y_2}(y_1, y_2) = \tau(y_2)\, e^{ik_1 y_1}.$$

Here, we have used the notation

$$k_1 = k\cos\theta, \qquad k_2 = k\sin\theta.$$

Notice that the first of the "monodromy" relations is satisfied by $w(y_1, y_2)$. Let $\mathbf{V}$ be the 2-vector

$$\mathbf{V}(y_2) = \begin{pmatrix} v(y_2) \\ \tau(y_2) \end{pmatrix}.$$

Then (4) reduces to

$$\frac{d\mathbf{V}}{dy_2}(y_2) = ik\mathbf{P}\mathbf{V}(y_2), \tag{6}$$

where $\mathbf{P}$ is the 2×2 matrix

$$\mathbf{P}(x_2) = \begin{bmatrix} 0 & 1/\mu \\ \rho\omega^2/k^2 - \mu\cos^2\theta & 0 \end{bmatrix}. \tag{7}$$

Note that $\mathbf{P}(y_2 + 1) = \mathbf{P}(y_2)$ by the periodicity of ρ and μ. The vector $\mathbf{V}$ must satisfy the second of the "monodromy" relations, which implies

$$\mathbf{V}(y_2 + 1) = \mathbf{V}(y_2)\, e^{ik_2}. \tag{8}$$

This equation is often referred to as the Floquet condition in the literature of ordinary differential equations.

We next study the system (6)-(7) for small values of k. Our starting point is the following *ansatz* for the frequency

$$\frac{\omega^2}{k^2} = \Omega_1^2 + (ik)\Omega_2 + (ik)^2\Omega_3 + \cdots. \tag{9}$$

The coefficient Ω_2 should be identically zero since it can be shown in general that the frequency is an even function of the wavenumber in periodic systems [2, 1]. We will naively ignore this fact and later deduce that Ω_2 does indeed vanish. Substituting (9) in (7), we get

$$\mathbf{P} = \mathbf{P}_0 + (ik)\mathbf{P}_1 + (ik)^2\mathbf{P}_2 + \cdots,$$

where

$$\mathbf{P}_0 = \begin{bmatrix} 0 & 1/\mu \\ -\mu\cos\theta + \rho\Omega_1^2 & 0 \end{bmatrix},$$

and

$$\mathbf{P}_1=\rho\Omega_2\mathbf{L},\qquad \mathbf{P}_2=\rho\Omega_3\mathbf{L},\quad \text{with } \mathbf{L}=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}.$$

Therefore, (6) becomes

$$\frac{d\mathbf{V}}{dx_2}=(ik\mathbf{P}_0+(ik)^2\mathbf{P}_1+(ik)^3\mathbf{P}_2+\cdots)\mathbf{V}.$$

We solve this by expanding $\mathbf{V}(x_2)$ in powers of (ik) and collecting terms of equal power. The solution is

$$\mathbf{V}(x_2)=\left\{\mathbf{I}+(ik)\int^{x_2}\mathbf{P}_0+(ik)^2\left[\int^{x_2}\mathbf{P}_0\int\mathbf{P}_0+\int^{x_2}\mathbf{P}_1\right]\right.$$
$$\left.+(ik)^3\left[\int^{x_2}\mathbf{P}_0\int\mathbf{P}_0\int\mathbf{P}_0+\int^{x_2}\mathbf{P}_0\int\mathbf{P}_1+\int^{x_2}\mathbf{P}_1\int\mathbf{P}_0+\int^{x_2}\mathbf{P}_2\right]\cdots\right\}\mathbf{V}(0). \tag{10}$$

Here, $\mathbf{I}=\mathrm{diag}(1,1)$ and

$$\int^{x_2}\mathbf{P}_0\equiv\int_0^{x_2}\mathbf{P}_0(y_1)\,dy_1,\qquad \int^{x_2}\mathbf{P}_0\int\mathbf{P}_0\equiv\int_0^{x_2}\mathbf{P}_0(y_1)\,dy_1\int_0^{y_1}\mathbf{P}_0(y_2)\,dy_2,$$

$$\int^{x_2}\mathbf{P}_0\int\mathbf{P}_0\int\mathbf{P}_0\equiv\int_0^{x_2}\mathbf{P}_0(y_1)\,dy_1\int_0^{y_1}\mathbf{P}_0(y_2)\,dy_2\int_0^{y_2}\mathbf{P}_0(y_3)\,dy_3,\qquad \ldots,\text{ etc.}$$

The Floquet condition (8) can be written as

$$\mathbf{V}(1)=\mathbf{V}(0)\left(1+ik\sin\theta+\frac{(ik)^2}{2}\sin^2\theta+\frac{(ik)^3}{6}\sin^3\theta+\cdots\right).$$

Substitution of (11) in the above leads to the equation

$$\det[\mathbf{S}_0+ik\mathbf{S}_1+(ik)^2\mathbf{S}_2+\cdots]=0, \tag{11a}$$

where $\mathbf{S}_j$, $j=0,1,2$, are 2×2 matrices involving $\mathbf{P}_k$, $k\leqslant j$,

$$\mathbf{S}_0=\int^1\mathbf{P}_0-\sin\theta\mathbf{I},\qquad \mathbf{S}_1=\int^1\mathbf{P}_0\int\mathbf{P}_0+\int^1\mathbf{P}_1-\frac{\sin^2\theta}{2}\mathbf{I}, \tag{11b, c}$$

and

$$\mathbf{S}_2=\int^1\mathbf{P}_0\int\mathbf{P}_0\int\mathbf{P}_0+\int^1\mathbf{P}_0\int\mathbf{P}_1+\int^1\mathbf{P}_1\int\mathbf{P}_0+\int^1\mathbf{P}_2-\frac{\sin^3\theta}{6}\mathbf{I}. \tag{11d}$$

We may simplify $\mathbf{S}_1$ by noting that

$$\int^1\mathbf{P}_0\int\mathbf{P}_0=\tfrac{1}{2}\left[\int^1\mathbf{P}_0\right]^2+\mathbf{T}, \tag{12a}$$

where

$$\mathbf{T}=\tfrac{1}{2}\left[\int^{1}\mathbf{P}_0\int\mathbf{P}_0-\left(\int^{1}\mathbf{P}_0^{\mathrm{T}}\int\mathbf{P}_0^{\mathrm{T}}\right)^{\mathrm{T}}\right]. \tag{12b}$$

Then eliminate the integral of $\mathbf{P}_0$ in (11c) in favor of $\mathbf{S}_0$ using (11b) and the definition of $\mathbf{P}_1$, to give

$$\mathbf{S}_1=\tfrac{1}{2}\mathbf{S}_0^2+\sin\theta\mathbf{S}_0+\mathbf{T}+\Omega_2\mathbf{L}\int^{1}\rho. \tag{13}$$

The equation (11a) can be expanded as a series in ik by using the expansion for the determinant of a 2×2 matrix,

$$\det(\mathbf{S}_0)+ik\,\mathrm{tr}(\hat{\mathbf{S}}_0\mathbf{S}_1)+(ik)^2[\mathrm{tr}(\hat{\mathbf{S}}_0\mathbf{S}_2)+\det(\mathbf{S}_1)]+\cdots=0, \tag{14}$$

where $\hat{\mathbf{S}}_0$ is the cofactor matrix of $\mathbf{S}_0$. The first order term in (14) implies that Ω_1 satisfies

$$\det\mathbf{S}_0=0. \tag{15}$$

Let $\langle\ \rangle$ denote the average over the unit cell, e.g. $\langle\rho\rangle=\int_0^1\rho(y)\,dy$. Then the effective medium condition (15) becomes

$$\langle\rho\rangle\Omega_1^2=\langle\mu\rangle\cos^2\theta+\left\langle\frac{1}{\mu}\right\rangle^{-1}\sin^2\theta. \tag{16}$$

The effective medium is therefore transversely anisotropic, with $c_{44}^*=c_{66}^*=\langle 1/\mu\rangle^{-1}$, $c_{55}^*=\langle\mu\rangle$, and $\rho^*=\langle\rho\rangle$.

The matrix $\mathbf{S}_0$ and its cofactor are of the form

$$\mathbf{S}_0=\begin{bmatrix}s_{11} & s_{12}\\ s_{21} & s_{11}\end{bmatrix},\qquad \hat{\mathbf{S}}_0=\begin{bmatrix}s_{11} & -s_{12}\\ -s_{21} & s_{11}\end{bmatrix},$$

where

$$s_{12}=\left\langle\frac{1}{\mu}\right\rangle.$$

Therefore,

$$\mathrm{tr}(\hat{\mathbf{S}}_0\mathbf{S}_1)=(s_{11}+2\sin\theta)\det(\mathbf{S}_0)+\mathrm{tr}(\hat{\mathbf{S}}_0\mathbf{T})-\Omega_2\langle\rho\rangle\left\langle\frac{1}{\mu}\right\rangle.$$

The first term on the right-hand side of this identity vanishes on account of (15), while the term $\mathrm{tr}(\hat{\mathbf{S}}_0\mathbf{T})$ vanishes because (12) implies that $\mathbf{T}$ is diagonal and traceless, and therefore the second term in (14) implies as expected [2, 1],

$$\Omega_2=0. \tag{17}$$

The $O(k^2)$ term in (14) simplifies considerably using $\Omega_2=0$. For instance, we have

$$\det \mathbf{S}_1=\det \mathbf{T}=-T_{11}^2 .$$

Using (11d), we find that

$$\Omega_3=\langle\rho\rangle^{-1}\left\langle\frac{1}{\mu}\right\rangle^{-1}\left[\tfrac{1}{3}\sin^4\theta+\operatorname{tr}\left(\hat{\mathbf{S}}_0\int_0^1\mathbf{P}_0\int\mathbf{P}_0\int\mathbf{P}_0\right)+\det\mathbf{S}_1\right].$$

After some algebraic manipulation, we obtain

$$-\langle\rho\rangle\left\langle\frac{1}{\mu}\right\rangle\Omega_3=\cos^4\theta\left[\langle\mu\rangle\left\langle\frac{1}{\mu}\right\rangle a+b_2^2+d-e\right]+2\sin^2\theta\cos^2\theta\left[a+b_1b_2-\tfrac{1}{2}\langle\mu\rangle^{-1}\left\langle\frac{1}{\mu}\right\rangle^{-1}e\right]$$
$$+\sin^4\theta\left[\langle\mu\rangle^{-1}\left\langle\frac{1}{\mu}\right\rangle^{-1}a+b_1^2-\tfrac{1}{3}\right], \tag{18}$$

where

$$a=\langle\rho\rangle^{-2}\langle\mu\rangle\left\langle\frac{1}{\mu}\right\rangle^{-1}\left[\left\langle\frac{1}{\mu}\right\rangle\int^1\rho\int\frac{1}{\mu}\int\rho+\langle\rho\rangle\int^1\frac{1}{\mu}\int\rho\int\frac{1}{\mu}\right],$$

$$b_1=\tfrac{1}{2}-\langle\rho\rangle^{-1}\left\langle\frac{1}{\mu}\right\rangle^{-1}\int^1\rho\int\frac{1}{\mu},\qquad b_2=\int^1\mu\int\frac{1}{\mu}-\langle\rho\rangle^{-1}\langle\mu\rangle\int^1\rho\int\frac{1}{\mu},$$

$$d=\left\langle\frac{1}{\mu}\right\rangle\int^1\mu\int\frac{1}{\mu}\int\mu+\langle\mu\rangle\int^1\frac{1}{\mu}\int\mu\int\frac{1}{\mu},$$

$$e=\langle\mu\rangle\int^1\frac{1}{\mu}\int\mu\int\frac{1}{\mu}+\langle\rho\rangle^{-1}\langle\mu\rangle^2\int^1\frac{1}{\mu}\int\rho\int\frac{1}{\mu}+\langle\rho\rangle^{-1}\langle\mu\rangle\left\langle\frac{1}{\mu}\right\rangle\left(\int^1\rho\int\frac{1}{\mu}\int\mu+\int^1\mu\int\frac{1}{\mu}\int\rho\right).$$

In summary, the desired dispersion relation $\omega(k_1,k_2)$ is

$$\omega^2(k_1,k_1)=k^2[\Omega_1^2(\theta)-k^2\Omega_3(\theta)+\cdots], \tag{19}$$

where Ω_1 and Ω_3 are given by eqs. (16) and (18). The dependence on k_1 and k_2 is through the variable θ,

$$\cos\theta=\frac{k_1}{k},\qquad \sin\theta=\frac{k_2}{k}.$$

In view of the representation for the leading order solution (5), we can interpret θ as the direction of a plane wave component whose wavenumber is (k_1,k_2).

A formula for Ω_3 for waves propagating in the direction normal to the layers is presented in Santosa and Symes [6]. This corresponds to the present case with $\theta=\pi/2$, and when this is substituted into (19) we find the resulting equation is in agreement with that in [6].

4. General properties and simplifications

4.1. The dispersion equation

Noting that the true dispersion is the one appearing in the exponential in (5), we now define

$$\bar{\omega}(k_1, k_2) = \frac{1}{h}\,\omega(hk_1, hk_2).$$

The dispersion relation $\bar{\omega}(k_1, k_2)$ depends upon the size of the basic period h and is given by

$$\bar{\omega}^2 = k^2(\Omega_1^2(\theta) - k^2h^2\Omega_3(\theta) + \cdots). \tag{20}$$

Within the asymptotic regime described in Section 2, small $\varepsilon = h/\lambda$ corresponds to small wave number k (see [6]), and consequently the frequency $\bar{\omega}$ is also small. These facts allow for easy manipulation of the dispersion relation. For instance, we can invert (20) to obtain

$$k^2 = \frac{1}{\Omega_1^2(\theta)}\,\bar{\omega}^2 + \frac{h^2\Omega_3(\theta)}{\Omega_1^6(\theta)}\,\bar{\omega}^4 + \cdots. \tag{21}$$

The form of the slowness diagram can be obtained from (21). Let $\mathbf{q} = (q_1, q_2)^{\mathrm{T}}$ be the slowness vector with

$$q_i = k_i/\bar{\omega}, \qquad i = 1, 2.$$

The magnitude of the slowness, $q = |\mathbf{q}|$, or $q = k/\bar{\omega}$, follows from (21),

$$q = \frac{1}{\Omega_1} + \frac{h^2\Omega_3}{2\Omega_1}\,\bar{\omega}^2 + \cdots.$$

For given values of h and $\bar{\omega}$ we define the slowness surface as the plot of q as a function of the angle of incidence θ. In the limit as $\bar{\omega} \to 0$, q is determined only by $\Omega_1(\theta)$ and the slowness surface is an ellipse (see Fig. 1). The same limit is achieved for $h \to 0$, which is the limit of zero cell size. Thus the formal equivalence of zero frequency limit and zero cell size limit is established. For $h \neq 0$ and $\bar{\omega} \neq 0$, the slowness surface is slightly deformed as shown in Fig. 1.

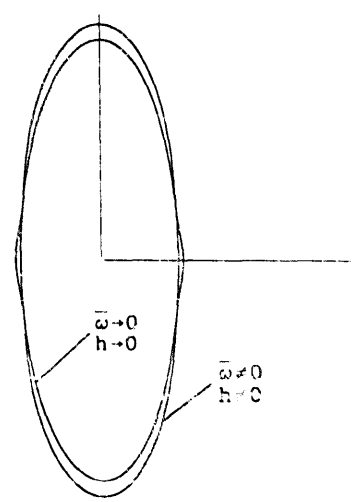

Fig. 1. Graphs of the slowness surfaces for dispersionless and dispersive propagations.

We next rewrite (20) in a form involving the two wavenumbers k_1 and k_2

$$\langle\rho\rangle\bar{\omega}^2 = C_{ij}k_ik_j - D_{ijmn}k_ik_jk_mk_n, \tag{22}$$

where the sum of repeated suffices is over 1 and 2, and the only non-zero elements C_{ij}, D_{ijkl}, are

$$C_{11} = \langle\mu\rangle, \qquad C_{22} = \left\langle\frac{1}{\mu}\right\rangle^{-1}, \tag{23a}$$

$$D_{1111} = h^2\left\langle\frac{1}{\mu}\right\rangle^{-1}\left[\langle\mu\rangle\left\langle\frac{1}{\mu}\right\rangle a + b_2^2 + d - e\right], \tag{23b}$$

$$D_{2222} = h^2\left\langle\frac{1}{\mu}\right\rangle^{-1}\left[\langle\mu\rangle^{-1}\left\langle\frac{1}{\mu}\right\rangle^{-1} a + b_1^2 - \tfrac{1}{3}\right], \tag{23c}$$

$$D_{1122} = D_{2211} = h^2\left\langle\frac{1}{\mu}\right\rangle^{-1}\left[a + b_1b_2 - \tfrac{1}{2}\langle\mu\rangle^{-1}\left\langle\frac{1}{\mu}\right\rangle^{-1} e\right]. \tag{23d}$$

The group velocity, defined as

$$\mathbf{c}_g = \nabla_k\bar{\omega}(k_1, k_2), \tag{24}$$

follows from (22) as

$$\langle\rho\rangle\bar{\omega}c_{gi} = C_{ij}k_j - 2D_{ijkm}k_jk_mk_n, \quad i = 1, 2.$$

Recalling that small ε corresponds to small k, we find that to leading order,

$$\langle\rho\rangle\bar{\omega}\mathbf{c}_g = \left(\langle\mu\rangle k_1, \left\langle\frac{1}{\mu}\right\rangle^{-1} k_2\right).$$

The phase direction is θ, where $\tan\theta = k_2/k_1$, and the direction of the group velocity vector is ϕ, where $\tan\phi = c_{g2}/c_{g1}$ or

$$\tan\phi = \langle\mu\rangle^{-1}\left\langle\frac{1}{\mu}\right\rangle^{-1}\tan\theta,$$

to first order. Since

$$\langle\mu\rangle\left\langle\frac{1}{\mu}\right\rangle \geqslant 1,$$

with the equality attained only if μ is constant, we deduce

$$\phi \leqslant \theta, \tag{25}$$

implying that the group velocity vector is always directed more towards the horizontal than the phase velocity vector.

4.2. The dynamic effective medium

The dispersion in (22) together with the representation in (5) suggests that the leading order solution $u(x_1, x_2, t)$ satisfies the partial differential equation

$$\langle\rho\rangle \frac{\partial^2 u}{\partial t^2} = C_{ij} \frac{\partial^2 u}{\partial x_i\, \partial x_j} + D_{ijkl} \frac{\partial^4 u}{\partial x_i\, \partial x_j\, \partial x_k\, \partial x_l}, \tag{26}$$

where again the sum on the repeated indices is over 1 and 2. It is tempting to use this equation in problems involving boundaries. However, this may be inappropriate because the equation is of order four in x, while a well-posed mixed boundary value problem for the SH waves involves only u and its normal derivative. This means that we have no information on the higher derivatives of u on the boundary, which are needed to solve initial boundary value problems for u satisfying (26).

We emphasize that this is an asymptotic theory. Keeping in mind that small ε is equivalent to small k, it is possible to write down other partial differential equations whose dispersion relations $\bar{\omega}^2(k_1, k_2)$ agree with that of (26) up to order k^4. For instance, one such equation is

$$\langle\rho\rangle \frac{\partial^2 u}{\partial t^2} + \langle\rho\rangle E_{ij} \frac{\partial^4 u}{\partial t^2\, \partial x_i\, \partial x_j} = C_{ij} \frac{\partial^2 u}{\partial x_i\, \partial x_j} + (D_{ijkl} + C_{ij} E_{kl}) \frac{\partial^4 u}{\partial x_i\, \partial x_j\, \partial x_k\, \partial x_l},$$

where $\mathbf{E}$ is an arbitrary 2×2 constant matrix. One could use this non-uniqueness to advantage. For example, $\mathbf{E}$ can be chosen so as to make the effective equation of motion strictly hyperbolic.

4.3. Constant impedance and constant wave speed

Define the impedance z and shear wave speed c,

$$z = \sqrt{\mu\rho}, \qquad c = \sqrt{\frac{\mu}{\rho}}.$$

The expression for Ω_3 simplifies somewhat if z is independent of x_2. Some of the double and triple integrals can be removed by the use of relations like

$$\int^1 f_1 \int f_2 \int f_3 = \langle f_1\rangle \int^1 f_2 \int f_3 - \int^1 f_2 \int f_1 \int f_3 - \int^1 f_2 \int f_3 \int f_1,$$

$$\int^1 f_1 \int f_2 = \langle f_1\rangle\langle f_2\rangle - \int^1 f_2 \int f_1,$$

where f_1, f_2 and f_3 are scalar functions. After a bit of simplification, we find

$$\begin{aligned} \Omega_3 = -\cos^4\theta \left\langle \frac{1}{c} \right\rangle^{-2} \Bigg[& \left\langle \frac{1}{c} \right\rangle \int^1 c \int \frac{1}{c} \int c + \langle c\rangle \int^1 \frac{1}{c} \int c \int \frac{1}{c} \\ & + \left(\int^1 \frac{1}{c} \int c \right)^2 - \langle c\rangle \left\langle \frac{1}{c} \right\rangle \int^1 \frac{1}{c} \int c - \tfrac{1}{12} \langle c\rangle^2 \left\langle \frac{1}{c} \right\rangle^2 \Bigg]. \end{aligned} \tag{27}$$

This is identically zero for waves propagating normal to the direction of layering ($\theta = \pi/2$). This is to be expected, since the lack of any impedance contrast between the layers means that no waves are reflected, and hence the signal travels without dispersion. If the wave propagates obliquely the reflection coefficients between layers are not zero, but depend upon the angles of incidence and transmission on either side of an interface. Hence, we would expect $\Omega_3 \neq 0$ for $\theta \neq \pi/2$, in general.

Alternatively, if c is constant, then the expression for Ω_3 bears a remarkable resemblance to (27)

$$\Omega_3 = -\sin^4\theta c^{-2}\langle z\rangle^{-3}\left\langle\frac{1}{z}\right\rangle^{-3}\left[\left\langle\frac{1}{z}\right\rangle\int^1 z\int\frac{1}{z}\int z+\langle z\rangle\int^1\frac{1}{z}\int z\int\frac{1}{z}\right.$$
$$\left.+\left(\int^1\frac{1}{z}\int z\right)^2-\langle z\rangle\left\langle\frac{1}{z}\right\rangle\int^1\frac{1}{z}\int z-\tfrac{1}{12}\langle z\rangle^2\left\langle\frac{1}{z}\right\rangle^2\right].$$

In this case the dispersion vanishes when the wave travels parallel to the layers, and is proportional to $\sin^4(\theta)$ otherwise. The fact that $\Omega_3(0)=0$ is not surprising considering that the wave travels in each layer with the same speed, and hence there is no tendency for the signal to spread out in the time domain.

4.4. *An alternative derivation via pseudo plane waves*

The equations of motion (3) can be transformed into a one-dimensional equation by taking the t- and the x_1-Fourier transform of (3). Thus, $U(k_1, x_2, \omega)$, the Fourier transform of $u(x_1, x_2, t)$, satisfies

$$\frac{\partial}{\partial x_2}\left(\mu\frac{\partial U}{\partial x_2}\right) = -(\rho\omega^2-\mu k_1^2)U.$$

Introduce the horizontal slowness p, defined by $k_1=\omega p$. The function $v(x_2, t; p)$ obtained by taking the inverse Fourier transform in ω of U is called a *pseudo plane wave*, and its one-dimensional equation of motion follows as

$$\frac{\partial}{\partial x_2}\left(\mu\left(\frac{x_2}{h}\right)\frac{\partial v}{\partial x_2}\right)=\left(\rho\left(\frac{x_2}{h}\right)-\mu\left(\frac{x_2}{h}\right)p^2\right)\frac{\partial^2 v}{\partial t^2}. \tag{28}$$

This decomposition is very much related to the Radon transform (see, for example, [9]), and leads to a popular computational method called the "reflectivity method" [10].

The value $p=0$ corresponds to wave propagation normal to the layers. We can write the dispersion relation for this particular case using (16), (19) with $\theta=\pi/2$ in (20).

$$\bar{\omega}^2 = q_0^{-2}k_2^2 - q_0^{-6}F[\rho]h^2k_2^4+\cdots \tag{29}$$

where the slowness q_0 is

$$q_0^2=\langle\rho\rangle\left\langle\frac{1}{\mu}\right\rangle. \tag{30}$$

The *functional* $F[\phi]$ in (29) operates on functions $\phi(x)$ defined on the unit interval, and its form follows from the equation for Ω_3 in (19). Thus

$$F[\phi]=-\left[\left\langle\frac{1}{\mu}\right\rangle\int^1\phi\int\frac{1}{\mu}\int\phi+\langle\phi\rangle\int^1\frac{1}{\mu}\int\phi\int\frac{1}{\mu}\right.$$
$$\left.+\left(\tfrac{1}{2}\langle\phi\rangle\left\langle\frac{1}{\mu}\right\rangle-\int^1\phi\int\frac{1}{\mu}\right)^2-\tfrac{1}{3}\langle\phi\rangle^2\left\langle\frac{1}{\mu}\right\rangle^2\right]. \tag{31}$$

Note that the dispersion relation (29) can be inverted to give

$$k^2=q_0^2\bar{\omega}^2+F[\rho]h^2\bar{\omega}^4+\cdots. \tag{32}$$

An arbitrary direction of propagation corresponds to $k_1=\bar{\omega}p$ and $k_2=\bar{\omega}p\tan\theta$. The analog to eq. (32) becomes

$$k_2^2=q(p)^2\bar{\omega}^2+A(p)h^2\bar{\omega}^4+\cdots \tag{33}$$

where $q(0)=q_0$ and $A(0)=F[\rho]$. By inspection of the one dimensional eq. (28) it is clear that $q(p)$ is obtained from (30) by replacing ρ with $\rho-\mu p^2$, and similarly

$$A(p)=F[\rho-\mu p^2]. \tag{34}$$

Making the substitution for $q(p)$ and expressing k_2 in terms of p in (33) implies

$$p^2=\frac{\cos^2\theta}{\Omega_1^2(\theta)}+A(p)\frac{\bar{\omega}^2h^2\cos^2\theta}{q_0^2\Omega_1^2(\theta)}+\cdots.$$

Iterating this equation yields an asymptotic expansion for p^2

$$p^2=\frac{\cos^2\theta}{\Omega_1^2(\theta)}+A\left(\frac{\cos\theta}{\Omega_1(\theta)}\right)\frac{\bar{\omega}^2h^2\cos^2\theta}{q_0^2\Omega_1^2(\theta)}+\cdots. \tag{35}$$

Recall that the asymptoptic expansion for $\bar{\omega}$ as a function of k is given by (20). The value of $\Omega_1(\theta)$ in this equation may be easily determined from static arguments, but suppose that we did not know $\Omega_3(\theta)$, except for the case of normal incidence $\theta=\pi/2$. A comparison of eqs. (20) and (29) implies that $\Omega_3(\pi/2)=q_0^{-6}F[\rho]$. Also, (20) may be rearranged into a form similar to (35) for arbitrary θ, viz

$$p^2=\frac{\cos^2\theta}{\Omega_1^2(\theta)}+\bar{\omega}^2h^2\cos^2\theta\,\frac{\Omega_3(\theta)}{\Omega_1^6(\theta)}+\cdots. \tag{36}$$

Comparison of (35) with (36), and use of (34) combined with the fact that the functional F is homogeneous and of second order in its argument, implies

$$\Omega_3(\theta)=q_0^{-6}F\left[\rho\sin^2\theta+\left\langle\frac{1}{\mu}\right\rangle(\rho\langle\mu\rangle-\langle\rho\rangle\mu)\cos^2\theta\right]. \tag{37}$$

It is then a simple matter of algebra to show that (31) and (37) are together equivalent to the previously determined explicit expression for Ω_3 in (19). Hence, if the form of the dispersion is known for waves traveling normal to the layers it may be used to generate the dispersion for oblique waves.

This alternative method of derivation may not always be applicable. For instance, in a related paper [8] on waves propagating normal to layers consisting of arbitrarily anisotropic elastic materials, Norris found the same functional F of (31), although in the fully elastic problem F operates on tensors. However, unlike the present situation, in the elastic problem there is no simple procedure like (29) by which one can simply use the results for normal propagation ($\theta=\pi/2$) to generate the dispersion for oblique propagation directions.

4.5. A three phase medium

The theory may be applied to composites with periodic layering of a finite number of different phases by first calculating the form of the functional F in (31) and then using the relation (37). The first task involves expressing the integrals in (31) as discrete sums. A procedure for accomplishing this is discussed by Norris [8] who provides an expression for F for a medium with four distinct layers in each unit cell. Here we will only consider two and three phase materials, with particular attention given to bilaminates.

Let the three phases have volume fractions n_1, n_2 and n_3 such that $n_1+n_2+n_3=1$. It may be shown [8] that $F[\rho]$ is

$$F[\rho]=\tfrac{1}{12}[n_1^2n_2^2\lambda_{12}^2+n_2^2n_3^2\lambda_{23}^2+n_3^2n_1^2\lambda_{31}^2-2n_1n_2n_3(n_1\lambda_{31}\lambda_{12}+n_2\lambda_{12}\lambda_{23}+n_3\lambda_{23}\lambda_{31})], \tag{38}$$

where

$$\lambda_{ij}=\frac{\rho_i}{\mu_j}-\frac{\rho_j}{\mu_i}.$$

The dispersion for oblique wave propagation follows from (37) and (38) as

$$\Omega_3(\theta)=\frac{1}{12q_0^6}[n_1^2n_2^2\gamma_{12}^2+n_2^2n_3^2\gamma_{23}^2+n_3^2n_1^2\gamma_{31}^2-2n_1n_2n_3(n_1\gamma_{31}\gamma_{12}+n_2\gamma_{12}\gamma_{23}+n_3\gamma_{23}\gamma_{31})], \tag{39}$$

where now

$$\gamma_{ij}=\langle\rho\rangle\left\langle\frac{1}{\mu}\right\rangle\left[\frac{\Omega_1^2(\theta)}{c_ic_j}\left(\frac{z_i}{z_j}-\frac{z_j}{z_i}\right)+\left(\frac{\mu_j}{\mu_i}-\frac{\mu_i}{\mu_j}\right)\cos^2\theta\right].$$

4.6. Two phases

The general results for a three phase medium simplify further for a two phase composite to give explicitly

$$\Omega_3(\theta)=\frac{n_1^2n_2^2}{12}\langle\rho\rangle^{-3}\left\langle\frac{1}{\mu}\right\rangle^{-3}\left[\frac{1}{c_1c_2}\left(\frac{z_1}{z_2}-\frac{z_2}{z_1}\right)\sin^2\theta+\left\langle\frac{1}{\mu}\right\rangle^2 z_1z_2\left(\frac{c_2}{c_1}-\frac{c_1}{c_2}\right)\cos^2\theta\right]^2. \tag{40}$$

It is clear from (40) that $\Omega_3>0$ for all θ, $0<\theta<\pi/2$, if $z_1\neq z_2$ and $(c_1-c_2)(z_1-z_2)<0$ If either (c_1-c_2) or (z_1-z_2) vanish then the dispersion also vanishes at the particular angles $\theta=0$ and $\theta=\pi/2$, respectively. These are just instances of the general simplifications that result when either the impedance or wave speed is constant, and discussed in the previous section. However, if

$$(c_1-c_2)(z_1-z_2)>0, \tag{41}$$

then Ω_3 vanishes at $\theta=\theta_0$, where

$$\tan\theta_0 = z_1 z_2 \left\langle \frac{1}{\mu} \right\rangle \left(\frac{c_1^2 - c_2^2}{z_1^2 - z_2^2} \right)^{1/2}, \tag{42}$$

and is strictly positive for all other θ, $0 \leqslant \theta \leqslant \pi/2$.

This angle has the following physical interpretation. Consider a ray that propagates successively through the alternating materials, with directions θ_1 and θ_2 in the respective layers, see Fig. 2, and related by Snell's law

$$\frac{\cos\theta_1}{c_1} = \frac{\cos\theta_2}{c_2}. \tag{43}$$

The associated reflection coefficient for a plane wave incident upon the interface between materials 1 and 2 is

$$R = \frac{z_1 \sin\theta_1 - z_2 \sin\theta_2}{z_1 \sin\theta_1 + z_2 \sin\theta_2}. \tag{44}$$

This vanishes if and only if the inequality (41) holds, and $\theta_1 = \theta_{10}$, $\theta_2 = \theta_{20}$, where

$$\tan\theta_{10} = \frac{z_2}{c_1} \left(\frac{c_1^2 - c_2^2}{z_1^2 - z_2^2} \right)^{1/2}, \qquad \tan\theta_{20} = \frac{z_1}{c_2} \left(\frac{c_1^2 - c_2^2}{z_1^2 - z_2^2} \right)^{1/2}, \tag{45}$$

which satisfy (43). Comparing (42) and (45) and using $\mu = zc$, we conclude that

$$\tan\theta_0 = n_1 \tan\theta_{10} + n_2 \tan\theta_{20}. \tag{46}$$

In general, we may define both θ_1 and θ_2 as functions of θ through (43) and the wavefront condition

$$\tan\theta = n_1 \tan\theta_1 + n_2 \tan\theta_2. \tag{47}$$

This ensures that the projection of the wavefront of the effective wave over a unit spatial period, $h \tan\theta$, is equal to the sum of the projections of the wavefronts in the distinct layers, $n_1 h \tan\theta_1$ and $n_2 h \tan\theta_2$, see

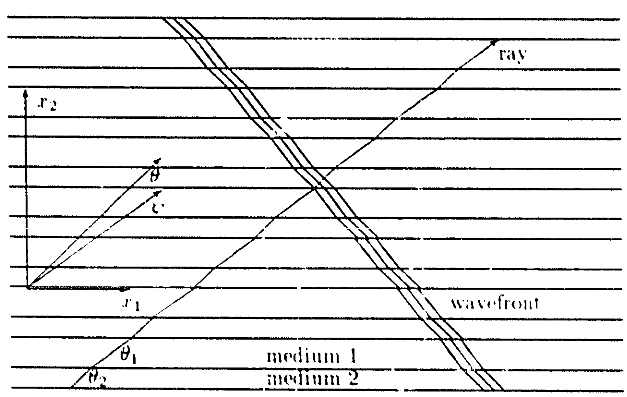

Fig. 2. The geometry of a ray and a wavefront through a layered periodic medium. The ray angles are θ_1 and θ_2 through the two phases. The wavefront is perpendicular to the ray. In this example, $\tan\theta_1 = 0.75$ and $\tan\theta_2 = 1$.

Fig. 2. It is then possible to write Ω_3 in a form involving θ_1 and θ_2,

$$\Omega_3(\theta)=\frac{n_1^2 n_2^2}{12}\langle\rho\rangle^{-3}\left\langle\frac{1}{\mu}\right\rangle^{-3}\cos^4\theta\left[\frac{z_1^2\sin^2\theta_1-z_2^2\sin^2\theta_2}{z_1 z_2 c_1 c_2}\right]^2$$
$$\times\left[\frac{n_1^2}{\cos^2\theta_1}+\frac{n_2^2}{\cos^2\theta_2}+\frac{2n_1n_2}{\cos\theta_1\cos\theta_2}\left(\frac{z_2\sin\theta_1+z_1\sin\theta_2}{z_1\sin\theta_1+z_2\sin\theta_2}\right)\right]^2.$$

Comparison of this with (44) implies directly that Ω_3 vanishes if and only if R vanishes, which occurs when $\theta_1=\theta_{10}$ and $\theta_2=\theta_{20}$, or equivalently, by (43) and (47), when $\theta=\theta_0$.

The existence of a non-dispersive direction when the inequality (41) holds has been previously noted for electromagnetic wave propagation through two phase layered media [11, 12]. In that context the effect is related to the Brewster angle phenomenon for perfect transmission, and as pointed out by Sipe et al. [11], it is not limited to periodic layerings, as long as the medium is composed of alternating materials. In the present context, we note that the effective medium is isotropic only if the shear modulus is constant. However, the inequality (41) cannot hold if the shear moduli are identical, because then

$$(c_1-c_2)(z_1-z_2)=-\frac{1}{c_1c_2}(c_1-c_2)^2.$$

Thus $\mu_1\neq\mu_2$ and the effective medium *must* be anisotropic for (41) to apply.

We note that the average direction, ψ, of the bilinear ray in Fig. 2 over many layers is given by

$$\cot\psi=n_1\cot\theta_1+n_2\cot\theta_2. \tag{48}$$

Equations (47), (48) and (43) imply that

$$\psi\leqslant\theta,$$

with equality only if the wavespeeds c_1 and c_2 are equal. The direction ψ therefore bears a similar qualitative relationship to the phase direction θ as the direction of the group velocity vector, ϕ, see (25). Although ψ and ϕ generally differ, they become the same when $\theta=\theta_0$, which is easily verified. This equality is not surprising, since at this phase angle the reflection coefficient R vanishes and therefore the energy of the effective wave should propagate in the kinematic direction ψ.

4.7. *An example*

A numerical example of a two phase material is now considered. Let the composite consist of materials with $\rho_1=\mu_1=1$, $\rho_2=2$, and $\mu_2=6$. The respective sound speeds are $c_1=1$ and $c_2=1.732$, while the acoustical impedances are $z_1=1$ and $z_2=3.464$. Hence, the product $(c_1-c_2)(z_1-z_2)>0$, and we would expect that $\Omega_3=0$ for some direction of incidence θ_0. For a volume fraction $n_1=0.4$, we use (42) to find

$$\theta_0=\frac{\pi}{5}.$$

This situation is illustrated in Fig. 3, where, as we have indicated, Ω_3 is nonnegative.

For the same material properties, we have computed Ω_3 for several angles of incidence θ as a function of volume fraction n_1. The graphs are shown in Fig. 4.

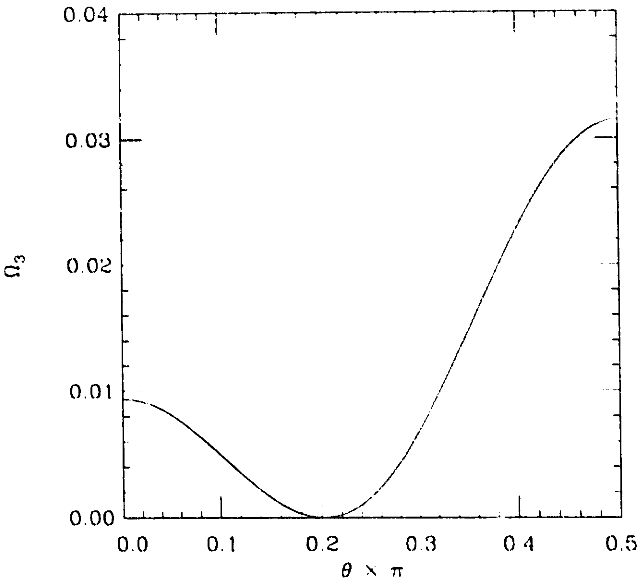

Fig. 3. Graph of Ω_3 for a two phase material as a function of angle of incidence. The material properties are $\rho_1 = \mu_1 = 1$, $\rho_2 = 2$, and $\mu_2 = 6$, with volume fraction $n_1 = 0.4$. Note that the dispersion coefficient is zero at $\theta_0 = \pi/5$ and maximum for propagation normal to the layers.

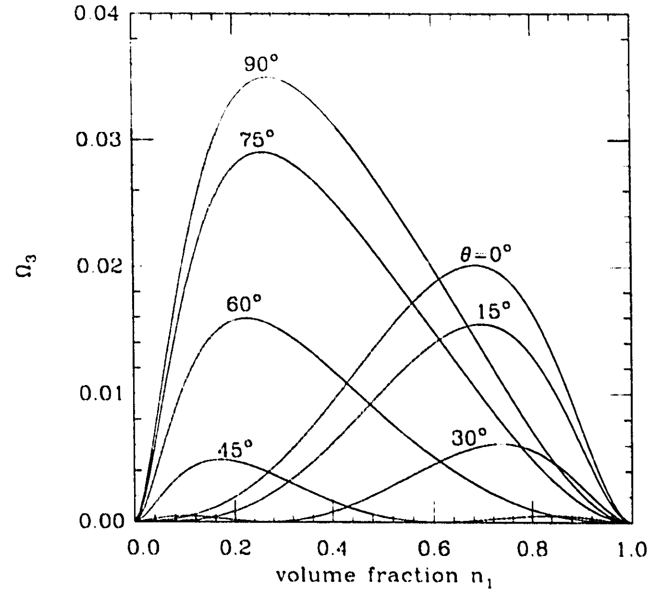

Fig. 4. Graph of Ω_3 for a two phase material as a function of volume fraction n_1 at several angles of incidence. The material properties are $\rho_1 = \mu_1 = 1$, $\rho_2 = 2$, and $\mu_2 = 6$.

4.8. Pseudo plane waves in a two phase medium

The relationship between the dynamic effective medium and the exact medium can be further understood using pseudo plane waves, introduced in eq. (28). In order that eq. (28) be hyperbolic we must have

$$\rho - \mu p^2 > 0.$$

This condition translates to a range on p. To be specific we consider a two phase composite, for which the hyperbolic condition becomes

$$p < \frac{1}{c_{\max}}, \quad \text{where } c_{\max} = \max(c_1, c_2).$$

A parameter value p satisfying the above inequality is said to be subcritical. The geometry of the ray and wavefront is identical to that in Fig. 2, where we make the identification

$$\tan\theta_1 = \frac{\sqrt{1 - c_1^2 p^2}}{c_1 p}, \qquad \tan\theta_2 = \frac{\sqrt{1 - c_2^2 p^2}}{c_2 p}.$$

We will solve initial value problems for $v(x_2, t; p)$ assuming that the initial data for v contain harmonic components whose wavelengths are no shorter than λ. In particular, we seek the asymptotic behavior of $v(x_2, t; p)$ for small $\varepsilon = h/\lambda$.

The effective medium equation for $v(x_2, t; p)$ is dispersive where the dispersion relation for small ε is of the form

$$\bar{\omega}^2 = \Omega_1^2 k_2^2 - \Omega_3 h^2 k_2^4 + \cdots . \tag{49}$$

This dispersion relation is obtained very simply by the procedure discussed in Section 4, where we used pseudo plane waves to rederive the general result for obliquely propagating waves. The effective soundspeed Ω_1 is found from (22) and the first of (23) to be

$$\Omega_1^2=\left\langle\frac{1}{\mu}\right\rangle^{-1}\langle\rho-\mu p^2\rangle^{-1}, \tag{50}$$

while the dispersion coefficient Ω_3 is determined from (40) for $\theta=\pi/2$,

$$\Omega_3=\frac{n_1^2n_2^2}{12\mu_1^2\mu_2^2}\langle\rho-\mu p^2\rangle^{-3}\left\langle\frac{1}{\mu}\right\rangle^{-3}[\mu_1(\rho_1-\mu_1p^2)-\mu_2(\rho_2-\mu_2p^2)]^2. \tag{51}$$

Both Ω_1 and Ω_3 are functions of the horizontal slowness p. Note that there is a value of p for which $\Omega_3=0$. This occurs when

$$p^2=\frac{\rho_1\mu_1-\rho_2\mu_2}{\mu_1^2-\mu_2^2}=\frac{z_1^2-z_2^2}{z_1^2c_1^2-z_2^2c_2^2}>0, \tag{52}$$

for which

$$(z_1-z_2)(z_1c_1-z_2c_2)>0,$$

is a necessary condition. The corresponding values of θ_1 and θ_2 are then identical to those of eq. (45) from which we see that the above inequality must be replaced by the more stringent (41) if both θ_1 and θ_2 are to be real, or equivalently, for the effective medium equation to maintain its hyperbolicity.

A ray traveling through the layers at slowness parameter p makes angles θ_1 and θ_2 in media 1 and 2 respectively. In the effective medium, the same ray makes an angle which can be found from the formula (50) for Ω_1. For k_2 small, the angle θ is

$$\cos\theta=\left[\left(\langle\mu\rangle\left\langle\frac{1}{\mu}\right\rangle-1\right)\frac{(\langle\rho\rangle-\langle\mu\rangle p^2)}{\langle\mu\rangle p^2}\right]^{1/2}.$$

Note that in the effective medium, a ray is subcritical when

$$\langle\rho\rangle-\langle\mu\rangle p^2>0.$$

Scrutiny of (51) leads to the conclusion that $\Omega_3\to\infty$ when p approaches the critical value defined by the effective medium.

We also note that the dispersion equation $\bar{\omega}^2(k_2)$ in (49) with Ω_1 and Ω_3 above can be found by a totally different method. Starting with the dynamic effective eq. (26), then taking Fourier transforms in t and x_1 and setting $k_1=\omega p$, gives

$$-\omega^2(\langle\rho\rangle-C_{11}p^2)U-D_{1111}\omega^4p^4U=C_{22}\frac{\partial^2U}{\partial x_2^2}-2D_{1122}\omega^2p^2\frac{\partial^2U}{\partial x_2^2}+D_{2222}\frac{\partial^4U}{\partial x_2^4}.$$

By letting k_2 be the wavenumber for the x_2 variable we obtain an equation satisfied by ω^2. For k_2 small, corresponding to our assumption that ε is small, the above equation yields a solution for ω^2 as a function of k_2 which agrees with $\bar{\omega}^2$ in (49) to $O(k_2^4)$. This equivalence is not surprising – it simply implies that the pseudo plane wave which satisfies (28) exhibits the same behavior as the pseudo plane wave for the effective medium satisfying (26).

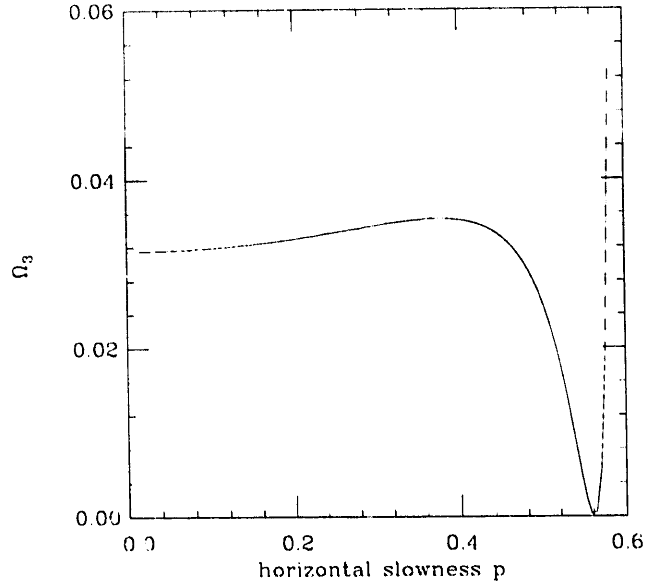

Fig. 5. The graph of Ω_3 as a function of p. The material properties are $\rho_1=\mu_1=1$, $\rho_2=2$, and $\mu_2=6$. The volume fraction is $n_1=0.4$.

In Fig. 5, we show a graph of Ω_3 as a function of p. The material properties are as before, $\rho_1=\mu_1=1$, $\rho_2=2$, and $\mu_2=6$, and the volume fraction is $n_1=0.4$. This graph exhibits a value p for which the dispersion coefficient is zero. The ray becomes critical in medium 2 at $p=0.577$ for this example, and the corresponding critical value in the effective medium is $p=0.632$. Note that Ω_3 becomes large as p approaches the latter value.

4.9 Shear wave propagation through anisotropic layers

The general theory is easily modified to account for material anisotropy in the layers. Specifically, we assume that each layer is composed of a material with orthorhombic symmetry such that the principal directions are aligned with the coordinate axes. The non-zero components of the stress tensor for SH motion in the x_3 direction are

$$\sigma_{31}=\sigma_{13}=c_{55}\left(\frac{x_2}{h}\right)\frac{\partial u}{\partial x_1}, \qquad \sigma_{32}=\sigma_{23}=c_{44}\left(\frac{x_2}{h}\right)\frac{\partial u}{\partial x_2}.$$

The moduli c_{44} and c_{55} are positive and become equal for isotropy. The matrix $\mathbf{P}$ is now

$$\mathbf{P}(x_2)=\begin{bmatrix} 0 & 1/c_{44} \\ \rho\omega^2/k^2-c_{55}\cos^2\theta & 0 \end{bmatrix}.$$

All of the asymptotic results for isotropic layers are easily modified for the anisotropic case by the substitutions $\mu\to c_{55}$ and $1/\mu\to 1/c_{44}$.

An example of practical importance is a bilaminate in which each layer is the same anisotropic material but with its axes of symmetry rotated relative to its neighbors. In fiber reinforced laminates each lamina can be viewed as a homogeneous but transversely isotropic material with axis of symmetry aligned with the fibers. If the laminae are arranged in a periodic manner then the overall laminate has orthorhombic symmetry. Furthermore, if the lay-up is /0/90/ then the moduli c_{44} and c_{55} alternate from layer to layer.

Thus, if the two shear moduli of the single transversely isotropic lamina are μ_a and μ_b, then for the /0/90/ lay-up there are two distinct types of layers, each with the same density ρ, with $c_{44}=\mu_a$, $c_{55}=\mu_b$ in one layer and $c_{44}=\mu_b$, $c_{55}=\mu_a$ in the other. The volume fractions are the same for both types of layers. The associated effective wave speed for the /0/90/ fiber reinforced bilaminate then follows from (16), using the prescription discussed above, as

$$\Omega_1^2=\frac{2\mu_a\mu_b}{\rho(\mu_a+\mu_b)}\left[1+\frac{(\mu_a-\mu_b)^2}{4\mu_a\mu_b}\cos^2\theta\right], \tag{53}$$

and the dispersion parameter follows from (40) as

$$\Omega_3=\frac{\rho\Omega_1^4}{96\mu_a\mu_b}\frac{(\mu_a-\mu_b)^2}{(\mu_a+\mu_b)}. \tag{54}$$

Alternatively, Ω_3 can be written using (53) and (54), as

$$\frac{\Omega_3}{\Omega_1^2}=\frac{1}{48}\left(\frac{\mu_a-\mu_b}{\mu_a+\mu_b}\right)^2\left[1+\frac{(\mu_a-\mu_b)^2}{4\mu_a\mu_b}\cos^2\theta\right].$$

This form indicates the relative magnitude of Ω_3 will always be small, and as a result dispersive effects will be significant only after propagation through many wavelengths. It is also clear that the dispersion parameter Ω_3 cannot vanish at any angle.

5. Accuracy of the asymptotic theory

In this section we provide details of a quantitative assessment of the accuracy of the asymptotic theory. In general the asymptotic solution to an initial value problem for (3) may be easily found in the following manner. Given the parameters of a periodic medium, first calculate the required dispersion relation in (20) and insert this into the representation as given by eq. (5). It is then a simple matter to obtain numerical values for the asymptotic solution. However, in order to gauge the accuracy of this solution we must have an exact solution with which to compare. Obtaining an exact solution is difficult in general. Accurate approximate solutions can be calculated by using difference schemes and taking care to make certain that the microstructure is adequately represented by choosing sufficiently small spatial grids. This is a challenging task in itself.

We choose to restrict the scope of our accuracy assessment by concentrating on a class of initial value problems whose solution can be obtained exactly. With the exact solution at hand, we can then proceed with the evaluation of the accuracy of the asymptotic solution. The class of initial value problems whose exact solutions can be found is the initial value problem for the pseudo plane wave. The particular initial value problem we consider is that of finding $v(x_2,t;p)$ for $t>0$ given that

$$\frac{\partial}{\partial x_2}\left(\mu\left(\frac{x_2}{h}\right)\frac{\partial v}{\partial x_2}\right)=\left(\rho\left(\frac{x_2}{h}\right)-\mu\left(\frac{x_2}{h}\right)p^2\right)\frac{\partial^2 v}{\partial t^2}, \tag{55}$$

with

$$\frac{\partial v}{\partial x_2}(x_2,0;p)=0 \quad \text{and} \quad v(x_2,0;p)=f(x_2).$$

The pseudo plane wave eq. (55) can be integrated exactly for piecewise constant density and shear modulus. We restrict ourselves further by considering only two phase materials. The calculation scheme used converts (55) to travel time coordinates. Once this is done, we discretize the computational domain, and a characteristic difference scheme is then derived. The scheme is *precise* if the jumps in the material properties occur at travel time nodes. The exact solution is piecewise constant in between the nodes. Therefore, the only limitation of the computational scheme we have designed is that the volume fraction must be chosen so that the jumps in the material properties occur at travel time nodes. A discussion of related schemes for Goupillaud layered media can be found in Section 3 of the review article [13].

Next consider the asymptotic solution for the pseudo plane wave (55). For a given microstructure, we calculate Ω_1 and Ω_3 using (50) and (51) respectively. The asymptotic solution is given by

$$v(x_2, t; p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)\exp(\mathrm{i}kx_2)\exp[\pm\mathrm{i}(\Omega_1 - h^2\Omega_3 k^2)kt]\,\mathrm{d}k,$$

with $F(k)$ being the Fourier transform of $f(x_2)$. In our comparison we will look at snapshots of $v(x_2, t; p)$ at a given t by calculating $v(x_2+\Omega_1 t, t; p)$. Our "camera" thus follows the right traveling pulse at speed Ω_1. Using an identity involving the Airy function, we can write

$$v(x_2+\Omega_1 t, t; p) = \tfrac{1}{2}\left(-\frac{\Omega_3 h^2 t}{2}\right)^{-1/3}\int_{-\infty}^{\infty} f(x_2 - x_2')\mathrm{Ai}\left(\left(-\frac{\Omega_3 h^2 t}{2}\right)^{-1/3} x_2'\right)\mathrm{d}x_2'. \tag{56}$$

See [6] for further details.

We will compare the snapshots of the exact solution to (55) with those of the asymptotic solution (56) for Gaussian initial displacement

$$v(x_2, 0; p) = \exp(-\alpha x_2^2).$$

We have set $\alpha = 0.138$ in all of the comparisons. The width of the Gaussian is defined as the size of the interval over which it is larger than 10^{-6}. Thus, the width of the Gaussian is about 20. The unit cell is fixed to be of length $h = 0.5$, implying that the width of the initial disturbances is about 40 times larger than the size of the microstructure.

The material properties are chosen to be the same as in the previous section, viz $\rho_1 = 1$, $\rho_2 = 2$, $\mu_1 = 1$, $\mu_2 = 6$, $c_1 = 1$, $c_2 = 1.732$, $z_1 = 1$ and $z_2 = 3.46$. The pseudo plane wave becomes critical in medium 2 when $p = 0.577$. We will look at several horizontal slowness values p and volume fractions, and in order to examine the effects of the dispersion, we will look at snapshots for $t \approx 1200$ and $t \approx 2400$.

The first numerical simulation is for $p = 0$ corresponding to normally incident waves. The volume fraction is $n_1 = 0.464$. We computed the effective properties using (50) and (51),

$$\Omega_1(p=0) = 1.085, \qquad \Omega_3(p=0) = 0.0195.$$

The value of Ω_3 suggests that dispersion effects will be strong only after large times, or large distances traveled by the pulse. At $t = 1200$, the center of the right-going pulse has traveled about 65 pulse widths. In Figs. 6(a) and 6(b), we display *both* the exact solutions and the asymptotic solutions. As can be seen, the agreement within this small aperture is very good. The disagreement is the small and rapid oscillation at the length scale of the microstructure that is visible in both figures. The asymptotic solution does not oscillate at the microstructural length scale. In Figs. 6(a) and 6(b) we have included the graphs of the undispersed initial pulse with its center located at the location predicted by the static effective medium theory ($x_2 = \Omega_1 t$). The dispersion is clearly substantial.

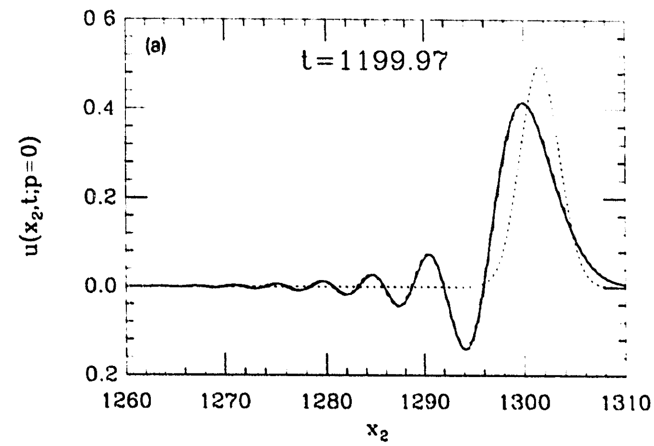

Fig. 6(a). The snapshot of $u(x_2, t; p)$ for $p=0$ (normal incidence) at $t=1199.97$, exact versus asymptotic theory. The only visible difference in the two graphs is the absence of rapid oscillations in the asymptotic solution. These oscillations are at the length scale of the microstructure. Shown in dashed line is the undispersed pulse as predicted by static theory.

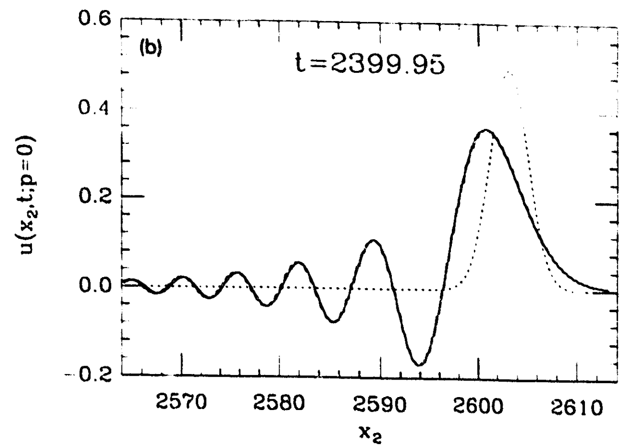

Fig. 6(b). The snapshot of $u(x_2, t; p=0)$ at $t=2399.95$, exact versus asymptotic theory. The dashed line is the prediction from static theory.

In the second trial we took $p=0.3$ with $n_1=0.437$, for which the computed effective properties are

$$\Omega_1(p=0.3)=1.243, \qquad \Omega_3(p=0.3)=0.0192.$$

The major difference as compared to previous examples is the increase in effective wavespeed. The center of the original disturbance has traveled about 75 pulse widths. The numerical results are displayed in Figs. 7(a) and 7(b). Again, we see good agreement between the asymptotic and the exact solutions.

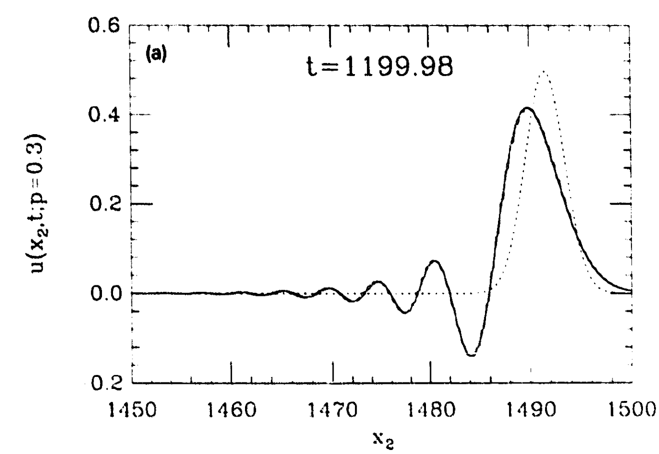

Fig. 7(a). The snapshot of $u(x_2, t; p)$ for $p=0.3$ at $t=1199.98$, exact versus asymptotic theory. Again, the only visible difference in the two graphs is the absence of rapid oscillations in the asymptotic solution. Shown in dashed line is the undispersed pulse as predicted by static theory.

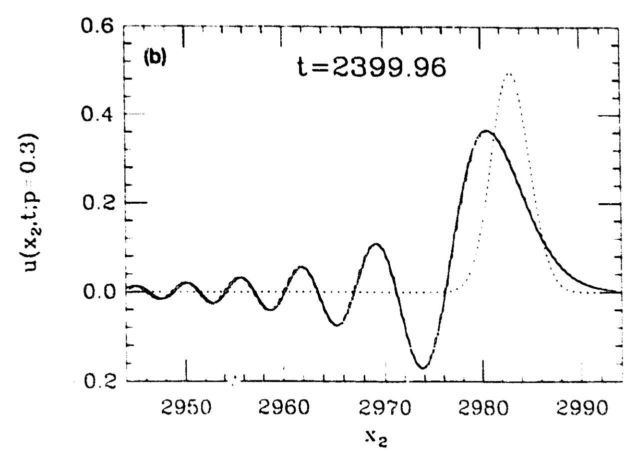

Fig. 7(b). The snapshot of $u(x_2, t; p=0.3)$ at $t=2399.96$, exact versus asymptotic theory. The dashed line is the prediction from static theory.

The last comparison is for $p=0.5$, corresponding to near critical in medium 2, and $n_1=0.333$. The effective properties are

$$\Omega_1(p=0.5)=1.964, \qquad \Omega_3(p=0.5)=0.0127.$$

The effective soundspeed is much larger in comparison to the first two cases, while the dispersion coefficient is smaller. At $t=1200$, the initial pulse has traveled about 117 pulse widths. Figures 8(a) and 8(b) compare the exact solution with the asymptotic solution at times $t=1200$ and $t=2400$, and once again the differences are insignificant except on length scales associated with the microstructure.

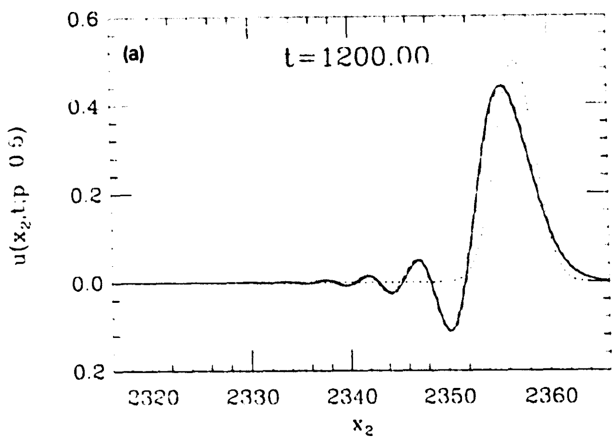

Fig. 8(a). The snapshot of $u(x_2, t; p)$ for $p=0.5$ (near critical in medium 2) at $t=1200$, exact versus asymptotic theory. The dashed line is the prediction from static theory.

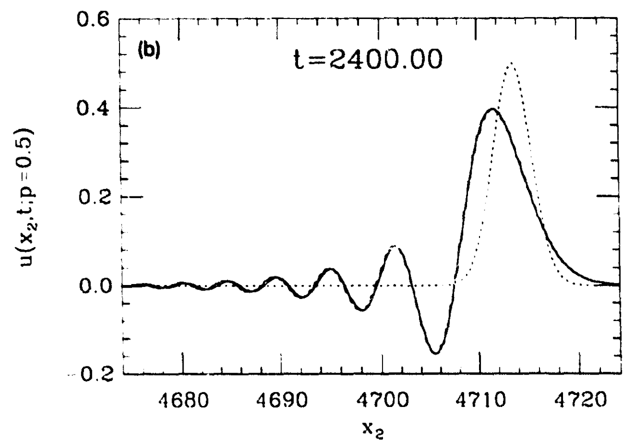

Fig. 8(b). The snapshot of $u(x_2, t; p=0.5)$ at $t=2400$, exact versus asymptotic theory. The dashed line is the prediction from static theory.

6. Conclusion

The numerical comparisons show quite convincingly that the dispersive effects are properly taken into account using the asymptotic theory. At the same time, it is clear from the computations that the simpler static effective medium theory is inadequate for treating propagation over large distances. Recall that the static effective medium replaces the periodic medium with a uniform anisotropic medium of density $\langle\rho\rangle$, and shear moduli $\langle 1/\mu\rangle^{-1}$ and $\langle\mu\rangle$. The *dynamic* effective medium, introduced in Section 4, would provide the same level of dispersion as the asymptotic theory for ω for relatively little additional effort.

The asymptotic series also facilitates the understanding of the relative influence of material parameters on the dispersion curves. Thus, we have seen that constant impedance and constant wavespeeds imply vanishing dispersion in the vertical and horizontal directions, respectively. The relative simplicity of the expression for Ω_3 for a two phase medium implies that it can vanish in a single direction if the inequality (41) holds, but is otherwise positive.

The analysis shown in this paper can be used to treat the problem of wave propagation in fully anisotropic layers. The number of dependent variables involved and the anisotropy of the medium make the calculation much more formidable [7, 8]. Future work will generalize the methods of this paper to other problems of waves in materials with periodic microstructure.

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