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Elastic Gaussian Wave Packets in Isotropic Media

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With 5 Figures

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Summary

The theory of Gaussian wave packet (GWP) propagation in elastic materials is developed. The GWP solutions are in the form of localized disturbances with Gaussian spatial envelopes at any instant in time. The method is explicitly time dependent, but is conceptually no more difficult than time harmonic ray theory. The equations of propagation and evolution are very similar to those of standard, elastodynamic ray theory, but include an extra degree of freedom not considered previously: the temporal width of the pulse. The theory is valid if the carrier wavelength is short in comparison with typical length scales in the medium. Interfaces of discontinuity in material properties give rise to reflected and transmitted GWPs. Explicit expressions are presented that relate the incident GWP to the reflected and transmitted GWPs. These results are illustrated by numerical simulations of a pulse incident upon a spherical interface.

1. Introduction

A time dependent theory is developed for the propagation of localized disturbances in elastic media. The elastic medium is assumed to be smoothly varying in space, except possibly at smooth surfaces of discontinuity. The shape of the disturbance is Gaussian in space at any instant in time. The Gaussian form is not initially assumed, but follows from the theory as the simplest type of localized disturbance. The related theory for the scalar wave equation is discussed by Norris, White and Schrieffer [1].

The present theory follows the asymptotic theory of time harmonic wave propagation in elastic media due to Karal and Keller [2]. However, the starting ansatz is now explicitly time dependent. In addition, it is assumed that the solution is of finite spatial extent. The center of the pulse follows the usual ray trajectories for longitudinal or transverse wave motions. The evolution of the Gaussian envelope is governed by a 3×3 complex analog of the real 2×2 wavefront curvature matrix of geometrical optics. This new matrix has not been considered previously, although it is related to 2×2 matrices used in the theory of time harmonic Gaussian beams. The latter theory has been discussed extensively in the context of the scalar wave equation [3], [4], [5]. Cerveny and Psencik [6] describe the theory of Gaussian beams in elastic media. An individual Gaussian beam is defined by a paraxial approximation of the phase about the central ray. The Gaussian wave packets in the present paper also involve a paraxial approximation in the ray direction itself. Thus, Gaussian beams may be viewed as special cases of Gaussian wave packets of infinite length.

The fundamental equations, starting ansatz and resulting sequence of asymptotic equations are discussed in Section 2. A wave packet traveling with the longitudinal wave speed is specifically considered in Section 3. A transverse GWP is considered in Section 4. The GWP solutions of Sections 3 and 4 use a local approximation to the phase in the neighborhood of the packet center that is correct to second order in space and time. The pre-exponential amplitude used is simply that at the packet center. It is shown in Section 5 that these two approximations are consistent. The procedure for extending the local approximations is outlined in Section 5. Section 6 considers the steps necessary for obtaining further terms in the asymptotic series solution. Finally, in Section 7, the reflection and transmission of GWPs at interfaces are discussed.

2. Equations of Motion

The equations of motion in a linearly elastic, isotropic but spatially inhomogeneous material are, in the absence of body forces,

$$(\lambda + \mu) \nabla\nabla \cdot \boldsymbol{u} + \mu\nabla^2\boldsymbol{u} + \nabla\lambda\nabla \cdot \boldsymbol{u} + \nabla\boldsymbol{u} \wedge (\nabla \wedge \boldsymbol{u}) + 2\nabla\mu \cdot \nabla\boldsymbol{u} - \varrho\boldsymbol{u}_{tt} = 0. \quad (2.1)$$

Here $\boldsymbol{u}(\boldsymbol{x}, t)$ is the displacement vector, $\lambda(\boldsymbol{x})$ and $\mu(\boldsymbol{x})$ the Lamé moduli, and $\varrho(\boldsymbol{x})$ the density. The subscript t denotes the partial time derivative. The starting point of the theory is the ansatz

$$\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{U}(\boldsymbol{x}, t)\, e^{i\omega\phi(\boldsymbol{x},t)} \quad (2.2)$$

$$\boldsymbol{U}(\boldsymbol{x}, t) = \sum_{n=0}^{\infty} (i\omega)^{-n}\, \boldsymbol{U}_n(\boldsymbol{x}, t). \quad (2.3)$$

This ansatz should be compared with that of Karal and Keller [2]. In their paper they made the time dependence explicitly harmonic by putting $\phi(\boldsymbol{x}, t) = S(\boldsymbol{x}) - t$. However, this assumption is not made here, and as we will see, the dependence of $\phi(\boldsymbol{x}, t)$ upon t will be more complicated than $\phi(\boldsymbol{x}, t) = S(\boldsymbol{x}) - t$.

Substituting (2.2) and (2.3) into (2.1) yields after some rearrangement of Serms

$$\begin{aligned}\sum_{n=0}^{\infty} (i\omega)^{-n} \{&[-\varrho \boldsymbol{U}_n\phi_t^2 + (\lambda + \mu)\, \nabla\phi(\boldsymbol{U}_n \cdot \nabla\phi) + \mu(\nabla\phi)^2\, \boldsymbol{U}_n] \\ &+ [-\varrho \boldsymbol{U}_{n-1}\phi_{tt} - 2\varrho \boldsymbol{U}_{n-1,t}\phi_t + (\lambda + \mu)\, \nabla(\boldsymbol{U}_{n-1} \cdot \nabla\phi) \\ &\quad + (\lambda + \mu)\, \nabla\phi(\nabla \cdot \boldsymbol{U}_{n-1}) + 2\mu(\nabla\phi \cdot \nabla)\, \boldsymbol{U}_{n-1} + \mu \boldsymbol{U}_{n-1}\nabla^2\phi \\ &\quad + \nabla\lambda(\boldsymbol{U}_{n-1} \cdot \nabla\phi) + \nabla\mu \wedge (\nabla\phi \wedge \boldsymbol{U}_{n-1}) + (2\nabla\mu \cdot \nabla\phi)\, \boldsymbol{U}_{n-1}] \\ &+ [-\varrho \boldsymbol{U}_{n-2,tt} + (\lambda + \mu)\, \nabla(\nabla \cdot \boldsymbol{U}_{n-2}) + \mu\nabla^2\boldsymbol{U}_{n-2} \\ &\quad + \nabla\lambda(\nabla \cdot \boldsymbol{U}_{n-2}) + \nabla\mu \wedge (\nabla \wedge \boldsymbol{U}_{n-2}) + 2(\nabla\mu \cdot \nabla)\, \boldsymbol{U}_{n-2}]\} = 0\end{aligned} \tag{2.4}$$

where by definition, $\boldsymbol{U}_{-1} = \boldsymbol{U}_{-2} = 0$. In the time harmonic case, $\phi(\boldsymbol{x}, t) = S(\boldsymbol{x}) - t$, and Eq. (2.4) reduces to Eq. (61) of Karal and Keller [2] (there is a mistake in the latter equation: the final $\nabla\lambda$ should be $\nabla\mu$). The asymptotic assumption $\omega \gg 1$, allows us to form a sequence of equations from (2.4) by equating to zero each coefficient of ω^{-n}, for $n = 0, 1, 2, \ldots$ The first equation for $n = 0$ is

$$c_L^2\nabla\phi(\nabla\phi \cdot \boldsymbol{U}_0) - c_T^2\nabla\phi \wedge (\nabla\phi \wedge \boldsymbol{U}_0) - \phi_t^2\boldsymbol{U}_0 = 0 \tag{2.5}$$

where

$$c_L(\boldsymbol{x}) = \left(\frac{\lambda + 2\mu}{\varrho}\right)^{1/2} \tag{2.6}$$

$$c_T(\boldsymbol{x}) = \left(\frac{\mu}{\varrho}\right)^{1/2}. \tag{2.7}$$

The speeds c_L and c_T are the longitudinal and transverse wave speeds. Eq. (2.5) is the elastodynamic eikonal equation. The necessary condition that a non-trivial $\boldsymbol{U}_0$ exists is either

$$\boldsymbol{U}_0 \parallel \nabla\phi \quad \text{and} \quad c_L^2(\nabla\phi)^2 = \phi_t^2 \tag{2.8.1, 2}$$

$$\boldsymbol{U}_0 \perp \nabla\phi \quad \text{and} \quad c_T^2(\nabla\phi)^2 = \phi_t^2. \tag{2.9.1, 2}$$

Denote the first case (2.8) as a longitudinal (L) solution, and the second case (2.9) as a transverse (T) solution). We will discuss each possibility separately.

Before proceeding to the longitudinal solution, we note the $n = 1$ or elastodynamic transport equation

$$\begin{aligned}&\varrho[c_L^2\nabla\phi(\boldsymbol{U}_1 \cdot \nabla\phi) - c_T^2\nabla\phi \wedge (\nabla\phi \wedge \boldsymbol{U}_1) - \phi_t^2\boldsymbol{U}_1] \\ &- \varrho \boldsymbol{U}_0\phi_{tt} - 2\varrho \boldsymbol{U}_{0t}\phi_t + (\lambda + \mu)\, \nabla(\boldsymbol{U}_0 \cdot \nabla\phi) + (\lambda + \mu)\, \nabla\phi(\nabla \cdot \boldsymbol{U}_0) \\ &+ 2\mu\nabla\phi \cdot \nabla\boldsymbol{U}_0 + \mu \boldsymbol{U}_0\nabla^2\phi + \nabla\lambda(\boldsymbol{U}_0 \cdot \nabla\phi) + \nabla\mu \wedge (\nabla\phi \wedge \boldsymbol{U}_0) + 2\boldsymbol{U}_0\nabla\mu \cdot \nabla\phi = 0.\end{aligned} \tag{2.10}$$

3. Longitudinal Gaussian Wave Packets

3.1 The Ray and Transport Equations

Define the unit vector

$$\boldsymbol{T} = \nabla\phi/|\nabla\phi|. \tag{3.1}$$

Then, by Eq. (2.8.1), define a scalar amplitude V_0,

$$\boldsymbol{U}_0 = V_0\boldsymbol{T}. \tag{3.2}$$

Substituting (3.2) into the transport Eq. (2.10) and taking the dot product of this with $\boldsymbol{T}$ gives, after some simple rearrangement,

$$\nabla \cdot [\varrho c_L^2 V_0^2 \nabla\phi] - \varrho\phi_t (V_0^2)_t - \varrho V_0^2 \phi_{tt} = 0. \tag{3.3}$$

The eikonal Eq. (2.8.2) has two roots. We consider explicitly the root

$$c_L\,|\nabla\phi| + \phi_t = 0 \tag{3.4}$$

but the other root can be treated by the same methods. The physical significance of the separate solutions corresponds to waves propagating in opposite directions. Using Eq. (3.4), Eq. (3.3) can be rewritten

$$\left[|\nabla\phi|\,\frac{\partial}{\partial t} + c_L\nabla\phi\cdot\nabla\right]\log\,(\varrho c_L V_0^2) + \left(\nabla\cdot c_L\nabla\phi - \frac{\phi_{tt}}{c_L}\right) = 0. \tag{3.5}$$

The ray paths are curves in space defined by the ray equations of (3.4),

$$\dot{\boldsymbol{x}}(t) = c_L\big(\bar{\boldsymbol{x}}(t)\big)\,\boldsymbol{p}(t)/|\boldsymbol{p}| \tag{3.6}$$

$$\dot{\boldsymbol{p}}(t) = -|\boldsymbol{p}|\;\nabla c_L\big(\bar{\boldsymbol{x}}(t)\big). \tag{3.7}$$

Here $\bar{\boldsymbol{x}}$ is the ray position and $\boldsymbol{p}$ the value of $\nabla\phi$ on the ray,

$$\boldsymbol{p}(t) = \nabla\phi\big(\bar{\boldsymbol{x}}(t), t\big). \tag{3.8}$$

The ray parameter t is the travel time on the ray. The dot over the quantities in (3.6) and (3.7) denotes the total derivative along the ray with respect to time. Alternatively, one could use arc length as the ray parameter. The relation between s and t is

$$s - s_0 = \int\limits_{t_0}^{t} c_L\big(\bar{\boldsymbol{x}}(t)\big)\,dt. \tag{3.9}$$

The ray is entirely specified by its initial position and direction.

Let $\bar{\phi}$ be the value of ϕ on the ray, $\bar{\phi} = \phi(\bar{\boldsymbol{x}}(t), t)$, then Eq. (3.4) implies

$$\dot{\bar{\phi}} = 0. \tag{3.10}$$

Thus, ϕ is constant along the ray. Note also, from (3.7), that

$$\dot{\overline{c_L\,|\boldsymbol{p}|}} = 0. \tag{3.11}$$

Therefore $c_L\,|\boldsymbol{p}|$ is constant on a ray. Without loss of generality, we may take the constant as unity, so that

$$|\boldsymbol{p}| = 1/c_L. \tag{3.12}$$

It then follows from (3.5) that along a ray,

$$\frac{d}{ds} \log\,(\varrho c_L V_0{}^2) + \left(\nabla \cdot c_L \nabla \phi - \frac{\phi_{tt}}{c_L}\right) = 0. \tag{3.13}$$

3.2 Ray Matrices and Their Ray Equations

Define the tangent vector

$$\boldsymbol{t} = \boldsymbol{p}/|\boldsymbol{p}| = \boldsymbol{T}(\bar{\boldsymbol{x}}(t), t) \tag{3.14}$$

and ray quantities

$$\boldsymbol{M}(t) = \nabla\nabla\phi(\bar{\boldsymbol{x}}(t), t) \tag{3.15}$$

$$\overline{V}_0(t) = V_0(\bar{\boldsymbol{x}}(t), t). \tag{3.16}$$

It follows from Eq. (3.4) that

$$\dot{\bar{\phi}}_t = 0 \tag{3.17}$$

on a ray. This result can be written, using (3.12), as

$$\bar{\phi}_{tt} = c_L{}^2\boldsymbol{t}^T\boldsymbol{M}\boldsymbol{t} + \boldsymbol{t} \cdot \nabla c_L. \tag{3.18}$$

Define the projection operator

$$\boldsymbol{P}_\perp = \boldsymbol{I} - \boldsymbol{t}\boldsymbol{t}^T. \tag{3.19}$$

Combining Eqs. (3.17), (3.18) and (3.19) gives

$$\frac{d}{ds} \log\,(\varrho c_L \overline{V}_0{}^2) + \mathrm{Tr}(c_L\,\boldsymbol{P}_\perp\boldsymbol{M}) = 0. \tag{3.20}$$

Now consider the matrix $\boldsymbol{A}(t)$ defined by

$$A_{ij}(t) = \frac{\partial \bar{x}_i(t)}{\partial \alpha_j}, \qquad i, j = 1, 2, 3 \tag{3.21}$$

where α is an arbitrary parametric vector independent of t, for example $\alpha = \bar{\boldsymbol{x}}(t_0)$, or $\alpha = \boldsymbol{p}(t_0)$. Similarly, define the matrix $\boldsymbol{B}(t)$ by

$$B_{ij}(t) = \frac{\partial p_i(t)}{\partial \alpha_j}, \qquad i, j = 1, 2, 3. \tag{3.22}$$

The evolution equations for $\boldsymbol{A}$ and $\boldsymbol{B}$ follow by differentiating (3.6) and (3.7), to give

$$\dot{\boldsymbol{A}} = \boldsymbol{t}(\nabla c_L)^T \boldsymbol{A} + c_L{}^2 \boldsymbol{P}_\perp \boldsymbol{B} \tag{3.23}$$

$$\dot{\boldsymbol{B}} = -\frac{1}{c_L} \boldsymbol{C}_L \boldsymbol{A} - (\nabla c_L)\, \boldsymbol{t}^T \boldsymbol{B} \tag{3.24}$$

where

$$(C_L)_{ij}(t) = \frac{\partial^2 c_L(\bar{\boldsymbol{x}}(t))}{\partial x_i \, \partial x_j}, \qquad i, j = 1, 2, 3. \tag{3.25}$$

Eqs. (3.23) and (3.24) along with

$$\boldsymbol{M}(t) = \boldsymbol{B}\boldsymbol{A}^{-1} \tag{3.26}$$

imply

$$\mathrm{Tr}\,(c_L \boldsymbol{P}_\perp \boldsymbol{M}) = \mathrm{Tr}\left(\frac{d\boldsymbol{A}}{ds}\boldsymbol{A}^{-1}\right) - \frac{1}{c_L}\frac{dc_L}{ds} = \frac{d}{ds}\log\left[\frac{\det(\boldsymbol{A})}{c_L}\right]. \tag{3.27}$$

Combining Eqs. (3.20) and (3.27) gives

$$\bar{V}_0(t) = \bar{V}_0(t_0)\left[\frac{\varrho(\bar{\boldsymbol{x}}(t_0)) \det \boldsymbol{A}(t_0)}{\varrho(\bar{\boldsymbol{x}}(t)) \det \boldsymbol{A}(t)}\right]^{1/2}. \tag{3.28}$$

3.3 A Local Approximation Leading to the Gaussian Wave Packet Solution

The first term in the asymptotic series (2.2) and (2.3) now follows from (3.2), (3.14), and (3.28)

$$\boldsymbol{U}(\bar{\boldsymbol{x}}(t), t) = \bar{V}_0(t)\, e^{i\omega\bar{\phi}(t_0)}\boldsymbol{t}. \tag{3.29}$$

This describes the solution at the ray position $\bar{\boldsymbol{x}}(t)$, which is determined by Eqs. (3.6) and (3.7), subject to initial conditions. At points in space and time near $(\bar{\boldsymbol{x}}(\tau), \tau)$, $\boldsymbol{U}(\boldsymbol{x}, t)$ may be approximated by taking the same pre-exponential amplitude, but using a local Taylor series approximation to $\phi(\boldsymbol{x}, t)$. Retaining terms up to second order in the latter gives,

$$\begin{aligned}\phi(\boldsymbol{x}, t) = {} & \phi(\bar{\boldsymbol{x}}, \tau) + (\boldsymbol{x} - \bar{\boldsymbol{x}}) \cdot \nabla\bar{\phi}(\tau) + (t - \tau)\, \bar{\phi}_t(\tau) \\ & + \frac{1}{2}(\boldsymbol{x} - \bar{\boldsymbol{x}})^T \nabla\nabla\bar{\phi}(\tau)\,(\boldsymbol{x} - \bar{\boldsymbol{x}}) + \frac{1}{2}(t - \tau)^2\, \bar{\phi}_{tt}(\tau) \\ & + (t - \tau)\,(\boldsymbol{x} - \bar{\boldsymbol{x}}) \cdot \bar{\nabla}\phi_t(\tau)\end{aligned} \tag{3.30}$$

where $\bar{\boldsymbol{x}} = \bar{\boldsymbol{x}}(\tau)$ is understood. This may be simplified by using (3.17), which gives $\bar{\phi}_t(\tau) = -1$. Also, (3.4) implies

$$\nabla \bar{\phi}_t(\tau) = -c_L \boldsymbol{M}(\tau)\, \boldsymbol{t} - \frac{\nabla c_L}{c_L}. \tag{3.31}$$

The quadratic paraxial approximation to the phase becomes, with $\Delta t \equiv t - \tau$ and $\Delta \boldsymbol{x} = \boldsymbol{x} - \bar{\boldsymbol{x}}(\tau)$,

$$\begin{aligned} \phi(\boldsymbol{x}, t) &= \phi(t_0) + \frac{1}{c_L}\, \boldsymbol{t} \cdot \Delta \boldsymbol{x} - \Delta t + \frac{1}{2}\, \Delta \boldsymbol{x}^T \boldsymbol{M}(\tau)\, \Delta \boldsymbol{x} \\ &\quad + \frac{1}{2}\, (\Delta t)^2 [c_L{}^2 \boldsymbol{t}^T \boldsymbol{M}(\tau)\, \boldsymbol{t} + \boldsymbol{t} \cdot \nabla c_L] - \Delta t\, \Delta \boldsymbol{x}^T \left[\frac{\nabla c_L}{c_L} + c_L \boldsymbol{M}(\tau)\, \boldsymbol{t}\right] \\ &= \phi(t_0) + \frac{1}{c_L}\, \boldsymbol{t} \cdot (\Delta \boldsymbol{x} - \boldsymbol{t} c_L\, \Delta t) \\ &\quad + \frac{1}{2}\, (\Delta \boldsymbol{x} - \boldsymbol{t} c_L\, \Delta t)^T\, \boldsymbol{M}(\tau)\, (\Delta \boldsymbol{x} - \boldsymbol{t} c_L\, \Delta t) \\ &\quad + \Delta t\, \frac{\nabla c_L}{c_L} \cdot \left[\boldsymbol{t}\, \frac{c_L}{2}\, \Delta t - \Delta \boldsymbol{x}\right]. \end{aligned} \tag{3.32}$$

This localized phase is then combined with the solution (3.29) on the ray, to give a GWP solution that is asymptotically valid in the neighborhood of the ray $\left(\bar{\boldsymbol{x}}(\tau), \tau\right)$,

$$\boldsymbol{U}(\boldsymbol{x}, t) = \boldsymbol{t} \overline{V}_0(\tau)\, e^{i\omega\phi(\boldsymbol{x},t)}. \tag{3.33}$$

Here, $\boldsymbol{t}$ is the ray direction, $\overline{V}_0(\tau)$ is given by Eq. (3.28) and $\phi(\boldsymbol{x}, t)$ by Eq. (3.32). The decay of the solution away from the center at $\left(\bar{\boldsymbol{x}}(\tau), \tau\right)$ is governed through (3.32) and (3.33) by the quantity $\phi(\boldsymbol{x}, t) - \phi(t_0)$. However, all the variables in this expression are implicitly real except for the 3×3 matrix $\boldsymbol{M}(\tau)$. Thus,

$$|\boldsymbol{U}(x, t)| = \left|\boldsymbol{U}\left(\bar{\boldsymbol{x}}(\tau), \tau\right)\right|\, e^{-\omega \operatorname{Im}[\phi(\boldsymbol{x},t) - \phi(t_0)]} \tag{3.34}$$

where, within the paraxial approximation (3.32)

$$\operatorname{Im}[\phi(\boldsymbol{x}, t) - \phi(t_0)] = \frac{1}{2}\, (\Delta \boldsymbol{x} - \boldsymbol{t} c_L\, \Delta t)^T \operatorname{Im} \boldsymbol{M}(\tau)\, (\Delta \boldsymbol{x} - \boldsymbol{t} c_L\, \Delta t). \tag{3.35}$$

3.4 Uniform Localization and Boundedness

In order that the paraxial approximation to $\phi(\boldsymbol{x}, t)$ gives a localized solution, we must have (1) that $\boldsymbol{M}(\tau)$ is symmetric for all τ. This is a natural condition since $\boldsymbol{M}$ is the matrix of second order spatial derivatives. The solution is in the form of a localized Gaussian if (2) $\operatorname{Im} \boldsymbol{M}(\tau)$ is positive definite. Examining (3.28), it is clear that (3) $\det \boldsymbol{A}(\tau) \neq 0$ is also necessary in order that the amplitude remains finite. In Ref. [1], the same set of coupled equations, (3.23) and (3.24), was derived in the context of the scalar wave equation. It was shown there that the necessary and sufficient conditions to ensure (1), (2) and (3) are simply that they be satisfied at t_0. The form of the evolution equations for $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{M} = \boldsymbol{B}\boldsymbol{A}^{-1}$ then guarantees their subsequent satisfaction.

Let us examine these initial conditions on $\boldsymbol{A}$ and $\boldsymbol{M}$. The condition $\det \boldsymbol{A}(t_0) \neq 0$ just means that the starting GWP has finite amplitude. We may, without loss of generality, set $\boldsymbol{A}(t_0) = \boldsymbol{I}$. Then, $\boldsymbol{B}(t_0) = \boldsymbol{M}(t_0)$ is by definition a complex matrix with positive definite imaginary part. At later times, $\boldsymbol{A}(t)$ will generally be complex, due to the explicit coupling in Eqs. (3.23) and (3.24). Note that the coefficients in these equations are all real. It is also shown in [1] that the matrix $\boldsymbol{M}(t)$ is the most important. In fact, we do not need $\boldsymbol{A}$ or $\boldsymbol{B}$; the amplitude and phase can be determined completely from $\boldsymbol{M}$. The equation for $\boldsymbol{M}$ follows from (3.23)—(3.26) as

$$\dot{\boldsymbol{M}}(t) = -\frac{1}{c_L} \boldsymbol{C}_L - c_L^2 \boldsymbol{M}\boldsymbol{P}_\perp \boldsymbol{M} - \boldsymbol{M}\boldsymbol{t}(\nabla c_L)^T - (\nabla c_L)\, \boldsymbol{t}^T \boldsymbol{M}. \tag{3.36}$$

3.5 Example: Propagation in a Uniform Medium

Explicit solutions for $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{M}$ in a uniform medium are given in Ref. [1]. For example, let $t_0 = 0$, $\bar{\boldsymbol{x}}(o) = 0$, $\boldsymbol{t} = \boldsymbol{k}$, the unit vector in the z-direction, and $\boldsymbol{M}(o) = \operatorname{diag}(a, a, b)$. At time τ, the ray is at $\bar{z} = c_L \tau$. The longitudinal displacement at neighboring positions $z = \bar{z} + \Delta z$, $\sqrt{x^2 + y^2} = r$, and times $t = \tau + \Delta t$, is

$$\boldsymbol{U}(\boldsymbol{x}, t) = \frac{\boldsymbol{k}\bar{V}_0(o)}{(1 + c_L a \bar{z})} \exp\left\{ i\omega \left[\frac{\Delta z}{c_L} - \Delta t + \frac{1}{2}\, \frac{a r^2}{(1 + c_L a \bar{z})} + \frac{b}{2} (\Delta z - c_L\, \Delta t)^2 \right] \right\}. \tag{3.37}$$

The half-width W of the GWP is defined by $r = W$ such that $|\boldsymbol{U}(\boldsymbol{x}, \tau)| / |\boldsymbol{U}(\bar{\boldsymbol{x}}(\tau)\, \tau)| = e^{-1}$. For simplicity, let $a = i\alpha$, where α is real and positive, then the half-width is

$$W = \left(\frac{2}{\omega}\right)^{1/2} \left(\frac{1}{\alpha^2} + c_L^2 \bar{z}^2\right)^{1/4}. \tag{3.38}$$

This is almost constant in the Fresnel or near field region, $\bar{z} \ll 1/\alpha c_L$. It increases as the square root of $\bar{z}$ in the Fraunhofer or far field region $\bar{z} \gg 1/\alpha c_L$. The pulse half-length L, on the other hand, remains constant for all $\bar{z}$ at

$$L = \left(\frac{2}{\omega \operatorname{Im} b}\right)^{1/2}. \tag{3.39}$$

4. Transverse Gaussian Wave Packets

Assume that

$$\boldsymbol{U}_0(\boldsymbol{x}, t) = V_0 \boldsymbol{Q} \tag{4.1}$$

where $\boldsymbol{Q}$ is a unit vector perpendicular to $\nabla\phi$. Substituting (4.1) into the transport Eq. (2.10), then taking the dot product of this equation with $\boldsymbol{Q}$, and using (2.9), gives

$$\nabla \cdot (\varrho c_T^2 V_0^2 \nabla\phi) - \varrho\phi_t(V_0^2)_t - \varrho V_0^2 \phi_{tt} = 0. \tag{4.2}$$

Note the similarity this equation has to (3.3). If we now consider the root

$$c_T |\nabla\phi| + \phi_t = 0 \tag{4.3}$$

to the eikonal Eq. (2.9), then there is a complete correspondence between the analysis in Section 3 and the solution to (4.2) and (4.3). In fact, all we need do is replace the subscript L by subscript T in Eqs. (3.3)—(3.28) and (3.30)—(3.35). Therefore the analysis will not be repeated. Note that Eq. (3.29) becomes, for the transverse wave solution,

$$\boldsymbol{U}\big(\overline{\boldsymbol{x}}(t), t\big) = \overline{V}_0(t)\, e^{i\omega\overline{\Phi}(t_0)} \boldsymbol{q}(t) \tag{4.4}$$

where $\boldsymbol{q}(t)$ is the value of $\boldsymbol{Q}$ on the ray,

$$\boldsymbol{q}(t) = Q\big(\overline{\boldsymbol{x}}(t), t\big). \tag{4.5}$$

The major difference between the transverse and longitudinal GWPs is in their polarization vectors, which are perpendicular and parallel to the ray direction $\boldsymbol{t}$, respectively. In order to obtain the evolution of the transverse direction vector $\boldsymbol{q}$, introduce the unit transverse vectors [6]

$$\boldsymbol{e}_1(t) = \boldsymbol{n} \cos\theta - \boldsymbol{b} \sin\theta \tag{4.6}$$

$$\boldsymbol{e}_2(t) = \boldsymbol{n} \sin\theta + \boldsymbol{b} \cos\theta. \tag{4.7}$$

Here $\boldsymbol{n}$ and $\boldsymbol{b}$ are the ray normal and binormal unit vectors, and

$$\theta(t) = \int_{s_0}^{s} \tau(s)\, ds + \theta_0 \tag{4.8}$$

where $\tau(s)$ is the ray torsion. The Frenet equations are then

$$\dot{\boldsymbol{e}}_1 = -c_T\varkappa \cos\theta\, \boldsymbol{t} \tag{4.9}$$

$$\dot{\boldsymbol{e}}_2 = -c_T\varkappa \sin\theta\, \boldsymbol{t} \tag{4.10}$$

$$\dot{\boldsymbol{t}} = c_T\varkappa(\cos\theta\, \boldsymbol{e}_1 + \sin\theta\, \boldsymbol{e}_2) \tag{4.11}$$

where $\varkappa$ is the ray curvature.

The system $(\boldsymbol{t}, \boldsymbol{e}_1, \boldsymbol{e}_2)$ forms a right handed triad of orthonormal vectors. Define the transverse unit displacement vector in this basis as

$$\boldsymbol{q}(t) = \boldsymbol{e}_1(t) \cos \psi + \boldsymbol{e}_2(t) \sin \psi \tag{4.12}$$

where $\psi = \psi(t)$. We now show that ψ is constant along the ray, and thus $\boldsymbol{q}$ rotates as $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ rotate. Consider the relation

$$\dot{\boldsymbol{q}}(t) = -c_T \varkappa \cos (\theta - \psi)\, \boldsymbol{t} + \boldsymbol{t} \wedge \boldsymbol{q} \dot{\psi} \tag{4.13}$$

which follows from the Frenet Eqs. (4.9) and (4.10). However, taking the dot product of the transport Eq. (2.10) with $\boldsymbol{t} \wedge \boldsymbol{q}$, using (4.1), (4.3) and (4.5), and evaluating it on the ray, gives

$$\boldsymbol{t} \wedge \boldsymbol{q} \cdot \dot{\boldsymbol{q}} = 0. \tag{4.14}$$

Eqs. (4.13) and (4.14) combined imply that $\dot{\psi} = 0$, which completes the proof.

In summary the transverse GWP in the neighborhood of $\left(\bar{\boldsymbol{x}}(\tau), \tau\right)$ is

$$\boldsymbol{U}(\boldsymbol{x}, t) = \boldsymbol{q}(\tau)\, \overline{V}_0(\tau)\, e^{i\omega\phi(\boldsymbol{x},t)}. \tag{4.15}$$

Here $\boldsymbol{q}$ is orthogonal to the ray direction and remains constant in the basis $(\boldsymbol{e}_1, \boldsymbol{e}_2)$, which rotates with the ray. The amplitude $\overline{V}_0$ and paraxial phase $\phi(\boldsymbol{x}, t)$ follow from the results in Section 3.

5. Local Corrections to the Gaussian Wave Packet Solution

The GWP solutions (3.33) and (4.15) use the amplitude and displacement direction for a point on the central ray, but use a paraxial approximation to the phase near the ray. Further local corrections could be implemented by (1) using a higher order Taylor series approximation to $\phi(\boldsymbol{x}, t)$, and (2) by locally expanding the pre-exponential term $\boldsymbol{U}_0(\boldsymbol{x}, t)$ about the ray. The question of obtaining the higher order terms $\boldsymbol{U}_n$, $n > 0$ of Eq. (2.3) is discussed in the next section.

To proceed with (1), one needs to calculate third order derivatives of ϕ on the ray, for example $\nabla\nabla\nabla\phi\left(\bar{\boldsymbol{x}}(\tau), \tau\right)$. If we define the third order tensors $A_{ijk}^{(2)} = \partial A_{ij}/\partial \alpha_\varkappa$ and $B_{ijk}^{(2)} = \partial B_{ij}/\partial \alpha_k$, then $\nabla\nabla\nabla\phi = (\boldsymbol{B}^{(2)} - \boldsymbol{M}\boldsymbol{A}^{(2)})\, \boldsymbol{A}^{-1}\boldsymbol{A}^{-1}$. Coupled evolution equations for $\boldsymbol{A}^{(2)}(t)$ and $\boldsymbol{B}^{(2)}(t)$ can be obtained from Eqs. (3.23) and (3.24) in the same way that these equations were derived from Eqs. (3.6) and (3.7). Thus, the procedure to improve on the paraxial approximation is conceptually simple, but very laborious. Further details are given in Appendix A.

In regards to (2), consider the expansion of $\boldsymbol{U}_0$ for the longitudinal GWP to first order in $\Delta\boldsymbol{x} = \boldsymbol{x} - \bar{\boldsymbol{x}}(\tau)$ and $\Delta t = t - \tau$. It follows from Eq. (3.2) that

$$\boldsymbol{U}_0(\boldsymbol{x}, t) = \boldsymbol{t}\overline{V}_0 + \boldsymbol{t}(\Delta\boldsymbol{x} \cdot \nabla\overline{V}_0 + \Delta t\overline{V}_{0t}) + \overline{V}_0(\Delta\boldsymbol{x} \cdot \nabla\boldsymbol{T} + \Delta t\boldsymbol{T}_t) \tag{5.1}$$

where $\overline{V}_0 = \overline{V}_0(\tau)$. The two bracketed terms on the right hand side of Eq. (5.1) are the corrections to $\boldsymbol{U}_0$ parallel and perpendicular to $\boldsymbol{t}$. The latter follows from Eqs. (3.1), (3.15), (3.19) and (3.31) as

$$\Delta \boldsymbol{x} \cdot \nabla \boldsymbol{T} + \Delta t \boldsymbol{T}_t = c_L \boldsymbol{P}_\perp \left[\boldsymbol{M}(\Delta \boldsymbol{x} - \boldsymbol{t} c_L \Delta t) - \Delta t \frac{\nabla c_L}{c_L} \right]. \tag{5.2}$$

The term $\overline{V}_{0t}$ in (5.1) can be related via Eq. (3.3) to quantities discussed in Section 3 plus the new quantity $\boldsymbol{t} \cdot \nabla \overline{V}_0$. Thus, the parallel correction for $\boldsymbol{U}_0$ depends upon $\nabla \overline{V}_0$, which has not been determined as yet. An evolution equation could be derived for $\nabla \overline{V}_0(t)$ by considering the variational equation of Eq. (3.3), in the same sense that (3.23) and (3.24) are variational equations of (3.6) and (3.7). However, the equation for $\nabla \overline{V}_0(t)$ would contain the tensors $\boldsymbol{A}^{(2)}$ and $\boldsymbol{B}^{(2)}$ discussed above. Thus, knowledge of the third derivatives of the phase are necessary for any consistent corrections to the GWP solution.

A similar argument can be made for corrections to $\boldsymbol{U}_0$ for the transverse GWP. Thus, to first order,

$$\boldsymbol{U}_0(\boldsymbol{x}, t) = \boldsymbol{q}\overline{V}_0 + \boldsymbol{q}(\Delta \boldsymbol{x} \cdot \nabla \overline{V}_0 + \Delta t \overline{V}_{0t}) + \overline{V}_0 \Delta \boldsymbol{x} \cdot \nabla \boldsymbol{q}. \tag{5.3}$$

Again, this requires $\nabla \overline{V}_0$, which must be obtained by solving an evolution equation involving the third spatial derivatives of ϕ. In conclusion, the GWP solutions of Sections 3 and 4 are entirely consistent with a quadratic approximation to the phase.

6. Higher Order Asymptotic Corrections

The GWP solutions of Sections 3 and 4 used only the first term $\boldsymbol{U}_0$ of the asymptotic sequence in Eq. (2.3). We now discuss the procedure for obtaining further terms. Consider the longitudinal solution of Section 3. Taking the cross product of Eq. (2.10) with $\boldsymbol{t}$ then using Eqs. (3.2) and (3.4), gives on the ray,

$$\begin{aligned} &\varrho \left(\frac{c_T^2}{c_L^2} - 1 \right) \boldsymbol{t} \wedge \boldsymbol{U}_1 + 2\varrho \overline{V}_0 \boldsymbol{t} \wedge \boldsymbol{t}_t + (\lambda + \mu)\, \boldsymbol{t} \wedge \nabla(\overline{V}_0\, |\nabla\phi|) \\ &+ \frac{2\mu}{c_L} \boldsymbol{t} \wedge (\boldsymbol{t} \cdot \nabla)\, \overline{V}_0 \boldsymbol{t} + \frac{\overline{V}_0}{c_L} \boldsymbol{t} \wedge \nabla\lambda = 0. \end{aligned} \tag{6.1}$$

Some simplification is obtained by using relations like (3.4), (3.15) and (3.31), to give

$$\boldsymbol{t} \wedge \boldsymbol{t}_t = -\boldsymbol{t} \wedge [\nabla c_L + c_L^2 \boldsymbol{M}\boldsymbol{t}] \tag{6.2}$$

$$\boldsymbol{t} \wedge \nabla(\overline{V}_0\, |\nabla\phi|) = \overline{V}_0 \boldsymbol{t} \wedge \boldsymbol{M}\boldsymbol{t} + \frac{1}{c_L} \boldsymbol{t} \wedge \nabla \overline{V}_0 \tag{6.3}$$

$$\boldsymbol{t} \wedge (\boldsymbol{t} \cdot \nabla)\, \overline{V}_0 \boldsymbol{t} = \overline{V}_0 c_L \boldsymbol{t} \wedge \boldsymbol{M}\boldsymbol{t}. \tag{6.4}$$

Eq. (6.1) becomes

$$\boldsymbol{t} \wedge \left[\varrho \left(\frac{c_T^2}{c_L^2} - 1 \right) \dot{\boldsymbol{U}}_1 - 2\varrho \overline{V}_0 (\nabla c_L + c_L{}^2 \boldsymbol{M}\boldsymbol{t}) + (\lambda + 3\mu)\, \overline{V}_0 \boldsymbol{M}\boldsymbol{t} \right. \\ \left. + \frac{(\lambda + \mu)}{c_L} \nabla \overline{V}_0 + \frac{\overline{V}_0}{c_L} \nabla \lambda \right] = 0. \tag{6.5}$$

Therefore,

$$\boldsymbol{U}_1 = \boldsymbol{W}_1 + \overline{V}_1 \boldsymbol{t} \tag{6.6}$$

where

$$\boldsymbol{W}_1 = c_L \nabla \overline{V}_0 - c_L{}^2 \overline{V}_0 \boldsymbol{M}\boldsymbol{t} + \frac{\overline{V}_0 c_L}{\lambda + \mu} (c_L{}^2 \nabla \varrho - 2\nabla \mu). \tag{6.7}$$

Eqs. (6.6) and (6.7) are analogous to Eqs. (126) and (127) of Karal and Keller [2].

Proceeding in the same fashion as Karal and Keller [2], an evolution equation for $\overline{V}_1(t)$ of Eq. (6.6) could be obtained by taking the scalar product of $\boldsymbol{t}$ with the $n = 2$ term in Eq. (2.4). Note, however, that the vector $\boldsymbol{W}_1$ in Eq. (6.7) contains the term $\nabla \overline{V}_0$. As discussed in the previous section, this vector can only be determined after we have obtained the third spatial derivatives of the phase ϕ. Therefore, higher order variational equations must be solved before one can proceed with the analysis.

This complication was overlooked by Karal and Keller [2]. They deduced a sequence of ray type equations for the amplitudes $\alpha_n = \overline{V}_n/|\nabla\phi|$. But the inhomogeneous terms in their equation contained the vectors $\nabla \alpha_{n-1}$. Thus, just as we require $\nabla \overline{V}_0$ to get $\boldsymbol{U}_1$, se they needed $\nabla \alpha_0$ to get α_1. However, they did not indicate how to obtain $\nabla \alpha_0$. It is clear from the discussion in the previous section that higher order spatial derivatives of the phase are necessary. Karal and Keller [2] did not emphasize this fundamental complication. They correctly pointed out that $\nabla^2 S$ (their phase is $S(\boldsymbol{x})$) is related to the ray tube area mapping along the ray, or wave front curvature. Higher spatial derivatives, e.g. $\nabla^4 S$, are also related to the wavefront geometry, but are not usually considered. But such wavefront parameters are a necessary prerequisite to obtaining further asymptotic terms. In consequence, the hierarchy of Eqs. (132)—(136) in Ref. [2] is not a closed system of equations. Similar conclusions apply to the higher order (in ω^{-n}) terms in the transverse GWP solution.

7. Reflection and Transmission at Interfaces

7.1 Jump Conditions

Consider a GWP of either L or T type, incident upon a smooth surface S across which the material properties are discontinuous. A similar problem was discussed by Norris, White and Schrieffer [1] for the scalar wave equation. The method of solution follows that in Ref. [1] very closely. Introduce reflected and transmitted GWPs on either side of S. The total solution must give continuous normal tractions and displacements on S. Following Achenbach et al. [7], it is clear that both L and T GWPs are reflected and transmitted. Thus, the various parameters for four additional GWPs need to be determined.

The necessary equations follow by asymptotic matching on S. By matching the phases on S about the point $\boldsymbol{x}_0$ on S where the central ray of the incident GWP strikes S and about the instant in time t_0 when the center of the GWP arrives there, we deduce jump conditions for all the elements of the $\boldsymbol{M}$ matrices. Details are given in Ref. [1]. Jump conditions on the slowness vectors $\boldsymbol{p}$ also come from the local phase matching (see Ref. [1]). The polarization vectors and pre-exponential amplitudes are deduced by satisfying the boundary conditions at $(\boldsymbol{x}_0, t_0)$. This results in a set of 4 simultaneous equations for the amplitudes, 2 in the case of an interface that is free of traction. We note that the local curvature of the interface enters into jump conditions for the $\boldsymbol{M}$ matrices, but not in the amplitude equations. Thus, the reflection and transmission coefficients follow from the canonical problem of a plane, time harmonic wave incident upon a flat interface separating half spaces.

Relatively compact equations are now presented for the reflected and transmitted $\boldsymbol{M}$ matrices. Define the interface locally about the point $\boldsymbol{x}_0$ at which the incident beam's central ray strikes S at $t = t_0$ as

$$\boldsymbol{x} = \boldsymbol{x}_0 + \xi_1 \boldsymbol{t}_1 + \xi_2 \boldsymbol{t}_2 + \sum_{j,l=1,2} \xi_j \xi_l D_{jl} \boldsymbol{t}_3 . \tag{7.1}$$

Here, $\{\boldsymbol{t}_1, \boldsymbol{t}_2, \boldsymbol{t}_3\}$ is a right-handed triad of orthonormal vectors such that $\boldsymbol{t}_3$ is the normal to S at $\boldsymbol{x}_0$. The 2×2 symmetric matrix $\boldsymbol{D}$ describes the local curvature of S, and ξ_1 and ξ_2 are local coordinates on S. Define a right-handed triad of orthonormal vectors $\{\boldsymbol{e}_1{}^{(\alpha)}, \boldsymbol{e}_2{}^{(\alpha)}, \boldsymbol{e}_3{}^{(\alpha)}\}$ for each of the incident ($\alpha = I$), reflected ($\alpha = R$) and transmitted ($\alpha = T$) GWPs. Following Ref. [1], specify $\boldsymbol{e}_3{}^{(\alpha)}$ as the unit vector in the ray direction, thus $\boldsymbol{e}_3{}^{(\alpha)} = \boldsymbol{p}^{(\alpha)}/|\boldsymbol{p}^{(\alpha)}|$, for each of $\alpha = I, R, T$. In general, there will be five sets of $\boldsymbol{e}$ vectors, one for I, and two each for R and T. The slowness vectors $\boldsymbol{p}^{(\alpha)}$ follow from Snell's law that $\boldsymbol{p}^{(\alpha)} \cdot \boldsymbol{t}_j = \boldsymbol{p}^{(I)} \cdot \boldsymbol{t}_j$, for $j = 1, 2$ plus the fact that $|\boldsymbol{p}^{(\alpha)}| = 1/c^{(\alpha)}$, where $c^{(\alpha)}$ is the wave speed of the associated GWP. Define the 2×2 matrices $\boldsymbol{Q}^{(\alpha)}$ by

$$Q_{ij}^{(\alpha)} = \boldsymbol{t}_i \cdot \boldsymbol{e}_j{}^{(\alpha)}, \qquad \alpha = I, R, T \tag{7.2}$$

and the 3×3 matrices $\boldsymbol{U}^{(\alpha)}$ by

$$\boldsymbol{U}^{(\alpha)} = \begin{bmatrix} \boldsymbol{Q}^{(\alpha)} & 0 \\ 0 & c^{(\alpha)} \end{bmatrix}, \qquad \alpha = I, R, T. \tag{7.3}$$

Then with obvous notation, with $\boldsymbol{M}^{(\alpha)}$ defined in the $\{\boldsymbol{e}^{(\alpha)}\}$ basis,

$$\boldsymbol{M}^{(\beta)} = \boldsymbol{U}^{(\beta)^{-1}} \boldsymbol{U}^{(I)} \boldsymbol{M}^{(I)} (\boldsymbol{U}^{(\beta)^{-1}} \boldsymbol{U}^{(I)})^T + \boldsymbol{C}_\perp{}^{(\beta)} + \boldsymbol{G}^{(\beta)}, \qquad \beta = T, R. \tag{7.4}$$

Here, $\boldsymbol{C}_\perp{}^{(\beta)}$ is a real matrix orthogonal to $\boldsymbol{e}_3{}^{(\beta)}$, which contains the surface curvature effects, and $\boldsymbol{G}^{(\beta)}$ is a real matrix that depends upon the local gradients of $c^{(I)}$ and $c^{(\beta)}$ near $\boldsymbol{x}_0$. These matrices are defined below. Eq. (7.4) summarizes similar results in Ref. [1] that were cast in the form of jump conditions but were left unsolved. Thus, Eq. (7.4) is an explicit statement of the implicit results in Ref. [1], and may be easily verified from there.

One immediate consequence of Eq. (7.4), not previously mentioned, is that the reflected and transmitted GWPs are localized. This follows from

$$\det \boldsymbol{U}^{(\alpha)} = c^{(\alpha)}\boldsymbol{t}_3 \cdot \boldsymbol{e}_3, \qquad \alpha = I, R, T \tag{7.5}$$

which is non-zero by assumption. Hence, for any real vector $\boldsymbol{v}$

$$\boldsymbol{v}^T(\operatorname{Im} \boldsymbol{M}^{(\beta)})\, \boldsymbol{v} = [\boldsymbol{v}^T \boldsymbol{U}^{(\beta)^{-1}} \boldsymbol{U}^{(I)}] \operatorname{Im} \boldsymbol{M}^{(I)} [\boldsymbol{v}^T \boldsymbol{U}^{(\beta)^{-1}} \boldsymbol{U}^{(I)}]^T > 0, \qquad \beta = R, T \tag{7.6}$$

and so $\operatorname{Im} \boldsymbol{M}^{(\beta)}$ is positive definite, implying localization.

The curvature matrix in Eq. (7.4) is for $\beta = R, T$:

$$\boldsymbol{C}_{\perp}^{(\beta)} = 2\left[\frac{\boldsymbol{t}_3 \cdot \boldsymbol{e}_3^{(I)}}{c^{(I)}} - \frac{\boldsymbol{t}_3 \cdot \boldsymbol{e}_3^{(\beta)}}{c^{(\beta)}}\right] \begin{bmatrix} \boldsymbol{Q}^{(\beta)^{-1}} & \boldsymbol{D} & \boldsymbol{Q}^{(\beta)^{-1T}} & 0 \\ & 0 & & 0 \end{bmatrix} \tag{7.7}$$

where the 2×2 matrix $\boldsymbol{D}$ is defined in Eq. (7.11).

The dependence upon the velocity gradients in Eq. (7.4) is

$$\boldsymbol{G}^{(\beta)} = \boldsymbol{U}^{(\beta)^{-1}} [\boldsymbol{H}^{(I)} - \boldsymbol{H}^{(\beta)}]\, \boldsymbol{U}^{(\beta)^{-1T}}, \qquad \beta = R, T \tag{7.8}$$

where

$$\boldsymbol{H}^{(\alpha)} = (\boldsymbol{e}_3^{(\alpha)} \cdot \nabla c^{(\alpha)})\, \boldsymbol{g}^{(\alpha)} \boldsymbol{g}^{(\alpha)T} - \boldsymbol{f}^{(\alpha)} \boldsymbol{g}^{(\alpha)T} - \boldsymbol{g}^{(\alpha)} \boldsymbol{f}^{(\alpha)T} \tag{7.9}$$

$$\boldsymbol{f}^{(\alpha)} = \begin{bmatrix} \boldsymbol{t}_1 \cdot \nabla c^{(\alpha)}/c^{(\alpha)} \\ \boldsymbol{t}_2 \cdot \nabla c^{(\alpha)}/c^{(\alpha)} \\ 0 \end{bmatrix} \tag{7.10}$$

$$\boldsymbol{g}^{(\alpha)} = \begin{bmatrix} \boldsymbol{t}_1 \cdot \boldsymbol{e}_3^{(\alpha)}/c^{(\alpha)} \\ \boldsymbol{t}_2 \cdot \boldsymbol{e}_3^{(\alpha)}/c^{(\alpha)} \\ -1 \end{bmatrix}, \qquad \alpha = I, R, T. \tag{7.11}$$

Note that $\boldsymbol{G}^{(\beta)} = 0$, $\beta = R, T$ for reflection/transmission between homogeneous media, while $\boldsymbol{C}_{\perp}^{(\beta)} = 0$, $\beta = R, T$ for reflection/transmission at a flat interface.

7.2 Example: Ultrasonic Transmission Through a Curved Interface

As an example, consider a hypothetical ultrasonic experiment in a water bath. A flat transducer of radius W_0 radiates a pulse of center frequency f_0. Thus, we take ω of Eq. (2.2) to be equal to $2\pi f_0$, and α of Eq. (3.38) such that $W_0 = W(\bar{z} = 0)$, or $\alpha = 1/\pi f_0 W_0^2$. The initial GWP then has half-width equal to the transducer radius. Let the pulse be initially of half-length equal to n wavelengths, then from Eq. (3.39), the parameter b in Eq. (3.37) is $b = i f_0/\pi c_L^2 n^2$.

To be more specific, let us choose the unit of length as 1 mm, and the unit of time as 1 μsec. Assuming a central frequency f_0 of 5 MHz, the nondimensional radial frequency is $\omega = 31.42$. Let the transducer radius be $W_0 = 5$ mm, so that $\alpha = 1/393$. The speed of sound in water is $c_L = 1.5$. Let the initial pulse half-length be 5 wavelengths, so that $b = i/35.3$. The form of the pulse passing a given point on the axis is thus $\cos(10\pi t) \exp(-t^2)$. Let the pulse be normally incident upon a sphere of aluminum of radius, 50 mm at a distance of 30 mm from the transducer. If the center of the pulse leaves the transducer at $t = 0$, it will reach the interface at $t = 20$. Subsequently, for $t > 20$, there will be a

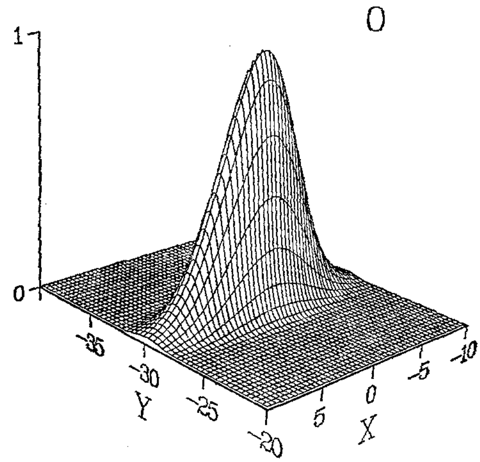

Fig. 1. The incident pulse at $t = 0$. The plot shows the envelope of the normal stress $\sigma_{22}(x, y, 0; t)$. The x and y units are in mm, and the stress is normalized to unity

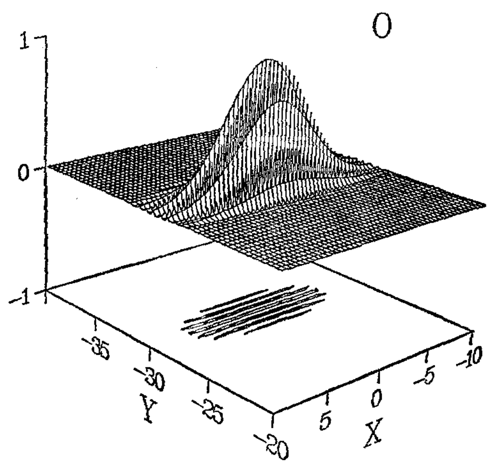

Fig. 2. The same as in Fig. 1, except that the phase factor has not been included. This shows the actual stress field, with rapid oscillations in the y-direction. These are evident from the projected contour lines

reflected and a transmitted pulse. Because of the normal incidence, no shear pulse is created in the aluminum.

Fig. 1 shows the envelope of the incident pulse at $t = 0$. The pulse is propagating in the positive y-direction. The envelope is defined by multiplying the solution in Eq. (3.33) by a phase factor $\exp(-iy/c_L)$. The inclusion of this factor eliminates the oscillations in the y-direction, and improves the visual appearance of the pulse. For comparison, Fig. 2 shows the same pulse with the oscillations present. Subsequent figures display pulse envelopes only. The transmitted pulse is shown in Fig. 3. Note the faster propagation in the metal, and the transverse oscillations in the pulses. These are caused mainly by the interfacial curvature. Since there are no focusing effects, the shape of the transmitted pulse does not change much between $t = 21$ and $t = 27$. However, focusing occurs if the interface curvature is reversed, i.e., if the incident pulse strikes a concave rather than a convex interface. This case is illustrated in Fig. 4, where we have considered a concave interface which is locally spherical of radius 50 mm

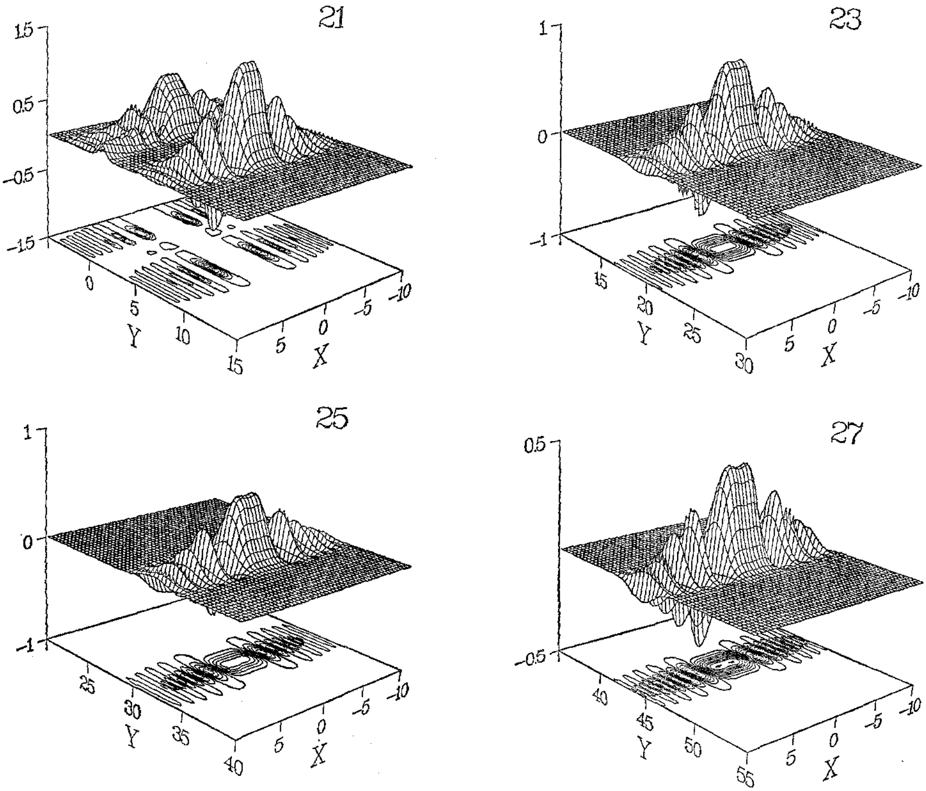

Fig. 3. The transmitted pulse in the aluminium sphere at times $t = 21$, 23, 25 and 27. The plots display the envelopes of the normal stress $\sigma_{22}(x, y, 0; t)$. At $t = 21$, the reflected pulse in the water is evident. The spherical interface $(y - 50)^2 + x^2 = (50)^2$ is shown at $t = 21$ by a low rim

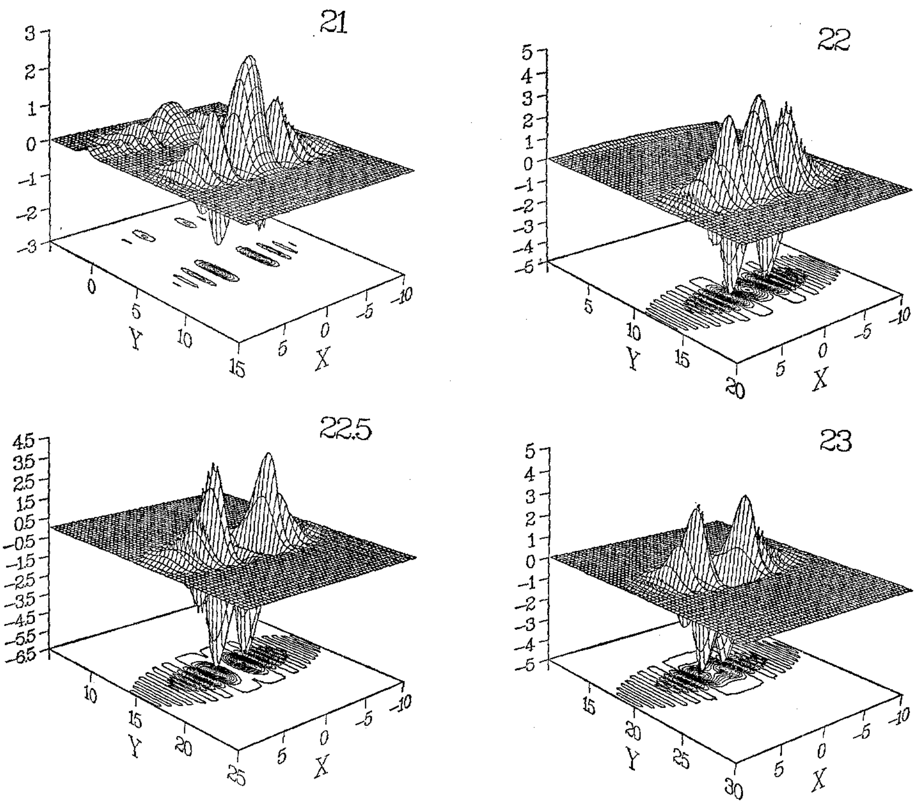

Fig. 4. The transmitted pulse for incidence upon a concave water/aluminum interface. The interface is spherical near the point of incidence of radius 50 mm. The times shown are $t = 21$, 22, 22.5 and 23. Note the larger amplitudes compared with Fig. 3.

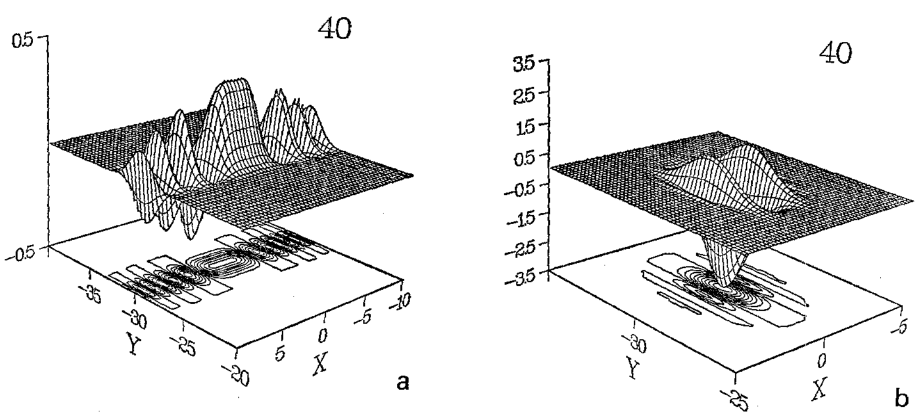

Fig. 5. The reflected pulses in the water at $t = 40$ for both the convex (a) and concave (b) interfaces considered in Figs. 3 and 4, respectively. Note the different (x, y) scales in (a) and (b)

at the point of incidence. A plane wave incident upon this interface has a transmitted focus at $t = 22.48$, i.e., 2.48 μsec after transmission. Note the radical change in shape of the pulses in Fig. 4 for $t \geqq 22.5$, and also the increased amplitude for $t \approx 22.5$.

Fig. 5 illustrates the reflected pulses in the water for both the convex and concave interfaces. In the latter case, it is clear from the figure that focusing occurs. The corresponding time at which a reflected plane wave focuses is $t = 47$.

8. Conclusion

The basic theory of GWP propagation is essentially just ray theory of elastodynamics plus ray-type equations for the matrix $\boldsymbol{M}$. The latter is the complex valued 3×3 envelope matrix, analogous to the real 2×2 wave front curvature matrix of geometrical optics. Explicit solutions for all the parameters can be obtained for homogeneous media. Very similar explicit solutions are also available for media in which the velocity gradients, ∇c_L and ∇c_T, are constant [1]. Piecewise homogeneous media can be considered by introducing reflected and transmitted GWPs. In general, the GWPs are uniform asymptotic solutions to the wave equations. Thus, unlike ordinary geometrical optics or geometrical elastodynamics, no boundary layer corrections are necessary at geometrical caustics and foci. This feature is one of the major advantages of GWPs. It makes them eminently suitable for use in modeling pulse propagation and scattering in elastic solids.

Appendix

Higher Order Spatial Derivatives of the Phase

Referring to Eqs. (3.6) and (3.7), they may be written as

$$\dot{X}_J(t) = F_J\big(\boldsymbol{X}(t)\big), \qquad J = 1, 6 \tag{A 1}$$

where $\boldsymbol{X}$ is a 6-vector with $\bar{\boldsymbol{x}}$ in its first 3 elements and $\boldsymbol{p}$ in the remaining elements. Similarly, $\boldsymbol{A}$ and $\boldsymbol{B}$ of (3.21) and (3.22) can be combined in the 6×3 matrix $\boldsymbol{G}^{(1)}$, where for $n \geqq 1$,

$$G^{(n)}_{Kj_1j_2\ldots j_n}(t) = \frac{\partial^n X_K(t)}{\partial \alpha_{j_1}\, \partial \alpha_{j_2} \ldots \partial \alpha_{j_n}}. \tag{A 2}$$

Here lower case subscripts have the values 1 to 3 and upper case subscripts the values 1 to 6. The evolution equations for $\boldsymbol{G}^{(n)}$ follow from (A 1) as

$$\dot{G}^{(n)}_{Ij_1j_2\ldots j_n}(t) = \frac{\partial F_I}{\partial X_K}\, G^{(n)}_{Kj_1j_2\ldots j_n} + H^{(n)}_{Ij_1j_2\ldots j_n} \tag{A 3}$$

where

$$H^{(n)}_{Ij_1j_2\ldots j_n} = \frac{\partial^n F_I}{\partial \alpha_{j_1}\,\partial \alpha_{j_2}\ldots\partial \alpha_{j_n}} - \frac{\partial \dot{F}_I}{\partial X_K}\,G_{Kj_1j_2\ldots j_n}. \tag{A 4}$$

Thus, $\boldsymbol{H}^{(n)}$ depends upon $\boldsymbol{G}^{(1)}, \boldsymbol{G}^{(2)} \ldots \boldsymbol{G}^{(n-1)}$ and derivatives of $\boldsymbol{F}$, but not upon $\boldsymbol{G}^{(n)}$. Eq. (A 3) represent a sequence of ordinary differential equations, which may be solved formally as

$$G^{(n)}_{Ij_1j_2\ldots j_n}(t) = g_{IJ}(t)\,G_{Jj_1j_2\ldots j_n}(o) + g_{IJ}(t)\int_0^t g^{-1}_{JK}(t')\,H^{(n)}_{Kj_1j_2\ldots j_n}(t')\,dt'. \tag{A 5}$$

Here $\boldsymbol{g}(t)$ is the fundamental matrix satisfying

$$\dot{g}_{IJ}(t) = \frac{\partial F_I}{\partial X_K}\,g_{KJ}(t) \tag{A 6}$$

$$g_{IJ}(o) = \delta_{IJ}. \tag{A 7}$$

Once the $\boldsymbol{G}^{(n)}$ are known, we may form the matrices

$$A^{(n)}_{ij_1j_2\ldots j_n} = \frac{\partial^n \bar{x}_i}{\partial \alpha_{j_1}\,\partial \alpha_{j_2}\ldots\partial \alpha_{j_n}} \tag{A 8}$$

$$B^{(n)}_{ij_1j_2\ldots j_n} = \frac{\partial^n p_i}{\partial \alpha_{j_1}\,\partial \alpha_{j_2}\ldots\partial \alpha_{j_n}}. \tag{A 9}$$

Thus, $\boldsymbol{A}^{(1)} = \boldsymbol{A}$, $\boldsymbol{B}^{(1)} = \boldsymbol{B}$. The higher order spatial derivatives of the phase,

$$M^{(n)}_{j_1j_2\ldots j_{n+1}} \equiv \frac{\partial^{n+1}\phi}{\partial x_{j_1}\,\partial x_{j_2}\ldots\partial x_{j_{n+1}}} \tag{A 10}$$

can then be found in terms of the $\boldsymbol{A}^{(n)}$ and $\boldsymbol{B}^{(n)}$. Thus,

$$M^{(1)}_{ij} = B^{(1)}_{ik}A^{(1)^{-1}}_{kj} \tag{A 11}$$

$$M^{(2)}_{ijk} = [B^{(2)}_{ipq} - M^{(1)}_{im}A^{(2)}_{mpq}]\,A^{(1)^{-1}}_{pj}A^{(1)^{-1}}_{qk} \tag{A 12}$$

etc.

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