

# Complex point-source representation of real point sources and the Gaussian beam summation method

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Received July 17, 1986; accepted July 23, 1986

A new exact representation for point sources is given in terms of complex point sources. In the simplest configuration, a point source is equivalent to a distribution of sources on the surface of a sphere in complex space. The representation can be used to consider the propagation of point disturbances through inhomogeneous media and across interfaces. In the high-frequency limit, these solutions may be obtained by the use of complex ray-tracing methods, which are just the analytic extension of ordinary ray methods. It is shown that the additional use of the paraxial approximation yields a procedure that is similar to the Gaussian beam summation method. The latter technique is normally based on a matched asymptotics argument. However, the complex point-source representation now offers an exact basis for this widely used method.

## INTRODUCTION

The use of complex point sources has received much attention recently because of its advantages over ordinary point sources in real space. Deschamps,<sup>1</sup> the originator of the concept of complex point sources, proposed them as an elegant and simple means of considering beams in electromagnetic wave theory. Other authors, particularly Felsen,<sup>2,3</sup> have explored the relation between complex point sources, evanescent waves, complex rays,<sup>4</sup> and Gaussian beams. All the classical solutions for real point sources, such as scattering from spheres and cylinders and diffraction from an edge,<sup>5</sup> can be converted quite simply into solutions for complex point sources by analytic continuation. These new solutions are particularly relevant to Gaussian beam phenomena. For example, it provides an exact method to study the Goos-Hänchen lateral beam shift of Gaussian beams at boundaries where the reflection coefficient varies with the angle of incidence.<sup>6</sup>

There has also been quite a bit of interest recently in Gaussian beams for quite different reasons.<sup>7,8</sup> The idea is to use Gaussian beams as an indirect means to compute the field of a point source in an inhomogeneous medium. It is well known that the field of a point source, considered in the context of geometrical optics in its simplest form, possesses singularities at caustics and foci. These singularities are an artifact of the asymptotic approximation employed and may be overcome by well-known but complicated boundary layer treatments. Individual Gaussian beams, on the other hand, are guaranteed to remain bounded at all points.<sup>8,9</sup> Also, they may be considered by using the same mathematical techniques of simple geometrical optics without recourse to boundary layer methods.<sup>9</sup> The key to the Gaussian beam summation method rests with the representation of the point source by an isotropic distribution of Gaussian beams all emanating from the same point. The amplitude of each beam is found by asymptotically matching the Gaussian beam field to that of a point source in the far field.

This paper describes a new method of representing real point sources through complex point sources. All the re-

sults are presented for a scalar, time-harmonic field but may be easily converted to the time domain. The simplest representation involves placing complex point sources of equal amplitude on the surface of a sphere in complex space. This representation is particularly relevant to the Gaussian beam summation method. It is shown how the latter follows from the exact representation in the high-frequency or short-wavelength limit, thus obviating the necessity of invoking matched asymptotics arguments.

The new representation can also be used to treat problems involving interfaces. The idea is to use complex ray-tracing techniques that satisfy Fermat's principle when either or both end points are in complex space. Application of this procedure yields the exact solution in a homogeneous medium, unlike the Gaussian beam method. A comparison of the two techniques is given for transmission across a spherical interface.

## REPRESENTATION INTEGRALS

The single-frequency, time-dependent factor  $\exp(-i\omega t)$  is assumed in what follows. The field at position  $\mathbf{x} = (x, y, z)$  due to a point source at position  $\mathbf{x}_0 = (x_0, y_0, z_0)$  in a homogeneous medium is

$$u^G(\mathbf{x}, \mathbf{x}_0) = \frac{e^{ikR}}{4\pi R}, \quad (1)$$

where  $k = \omega/c$  is the wave number and  $R$  is the distance between  $\mathbf{x}_0$  and  $\mathbf{x}$ . The function  $u^G$  is the fundamental solution or the Green function of the Helmholtz equation,

$$\nabla^2 u^G + k^2 u^G = -\delta(\mathbf{x} - \mathbf{x}_0). \quad (2)$$

A complex point source<sup>1</sup> is defined by letting  $\mathbf{x}_0$  be complex, i.e., each of its three components can be a complex number. The distance  $R$  is then defined as the complex number,

$$R(\mathbf{x}, \mathbf{x}_0) = [(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2]^{1/2}, \quad (3)$$

where the branch of the square root is specified by requiring that  $\text{Re } R \geq 0$ . For example, if  $\mathbf{x}_0 = (ib, 0, 0)$ ,  $b > 0$ , then  $-b$

$\leq \text{Im } R \leq b$ , and  $R$  and  $u$  are continuous everywhere in real space except across the disk  $x = 0, y^2 + z^2 \leq b^2$ . In the limit of  $b \rightarrow 0$ , the complex point source reduces to a real point source at the origin.

Let  $(r, \theta, \phi)$  be spherical coordinates with respect to the origin, and let  $\mathbf{e}_r$  be the unit vector in the radial direction. Then for any real number  $a$ , it follows that for  $r > a$ ,

$$\frac{1}{4\pi j_0(ka)} \int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta u^G(\mathbf{x}, a\mathbf{e}_r) = u^G(\mathbf{x}, 0), \quad (4)$$

where  $j_0(x) = (\sin x)/x$  is the spherical Bessel function of order zero. Equation (4) follows from symmetry considerations and from the fact that both sides are outgoing solutions to the Helmholtz equation in  $r > a$ . The constant of proportionality in Eq. (4) can be deduced by equating both sides as  $r \rightarrow \infty$ . The equations analogous to Eqs. (1) and (4) in two dimensions (no  $z$  or  $\theta$  dependence) are

$$u^G(\mathbf{x}, \mathbf{x}_0) = \frac{-i}{4} H_0^{(1)}(kr), \quad (5)$$

$$\frac{1}{2\pi J_0(ka)} \int_0^{2\pi} d\phi u^G(\mathbf{x}, a\mathbf{e}_r) = u^G(\mathbf{x}, 0). \quad (6)$$

These results, Eqs. (4) and (6), may be generalized to inhomogeneous media that are homogeneous for  $r \leq a$ . The point-source solutions  $u^G$  of Eq. (2) that are used in Eqs. (4) and (6) are now different from those given in Eqs. (1) and (5). In order to prove the generalization, define the Green function  $u_0^G(\mathbf{x}, \mathbf{x}_0)$  as the solution to Eq. (2) for  $\mathbf{x}$  and  $\mathbf{x}_0$  inside  $r = a$ , and such that  $u_0^G$  is zero on  $r = a$ . It then follows from Green's theorem that at any point  $\mathbf{x}_0$  inside  $r = a$ ,

$$u(\mathbf{x}_0) = - \int_S u(\mathbf{x}') \frac{\partial u_0^G}{\partial n}(\mathbf{x}', \mathbf{x}_0) dS(\mathbf{x}'), \quad (7)$$

where  $S$  is the surface  $r = a$  and  $n$  is the outward normal. Now, since the medium inside  $r = a$  is assumed homogeneous, we can solve for  $u_0^G$  by standard methods. In particular, for any  $\mathbf{x}$  on  $S$ ,

$$\frac{\partial u_0^G}{\partial n}(\mathbf{x}, 0) = \begin{cases} -[4\pi a^2 j_0(ka)]^{-1} & \text{in 3D} \\ -[2\pi a J_0(ka)]^{-1} & \text{in 2D} \end{cases}. \quad (8)$$

Now let  $u(\mathbf{x}') = u^G(\mathbf{x}', \mathbf{x})$  in Eq. (7), where  $\mathbf{x}$  is outside  $r = a$ . Then Eqs. (4) and (6) follow from Eqs. (7) and (8), and the reciprocity relation  $u^G(\mathbf{x}, \mathbf{x}_0) = u^G(\mathbf{x}_0, \mathbf{x})$ .

Now let the "radius"  $a$  in Eqs. (4) and (6) be complex. Then, by analytic continuation, these identities imply integral representations of real point sources in terms of complex point sources. Thus the field at  $\mathbf{x}$  due to a source at  $\mathbf{x}_0$  is equivalent to a distribution of complex sources on the surface of the complex sphere (circle) of radius  $a$  about  $\mathbf{x}_0$ . It also follows from reciprocity that the field at  $\mathbf{x}$  due to a source at  $\mathbf{x}_0$  is equivalent to the sum, with suitable weighting, of the fields at points on the surface of the complex sphere (circle) of radius  $a$  about  $\mathbf{x}$ . For example, let  $a = ib$ ,  $b$  real. Then each complex source on the surface of radius  $a$  about  $\mathbf{x}_0$  forms the sphere of radius  $b$  about  $\mathbf{x}_0$ . Therefore the representation integrals (4) and (6) are valid in real space for  $r > |b|$ . In general, when  $a$  is any complex number, Eqs. (4) and (6) are valid for  $r > |a|$ . Note that Eqs. (4) and (6) can be viewed as mean value theorems, which state that the

average of the field on the surface of a sphere (circle) of radius  $a$  is just  $j_0(ka)[J_0(ka)]$  times the field at the center.

If the radius  $a$  in Eq. (4) is real, then as  $k$  varies, the integral must become infinite at  $k = n\pi/a$ , the roots of  $j_0(ka) = 0$ . This instability problem is not present for a complex. Hence the representation will be stable for all frequencies.

Other representations may also be considered. For example, one could place point sources on the surface of a complex ellipsoid. However, the amplitude of each complex point source must vary with position if the ensemble is to be equivalent to a single point source in real space. The amplitudes can be found by solving an integral equation over the surface of the ellipsoid.

## THE GAUSSIAN BEAM SUMMATION METHOD

For simplicity, consider the field at  $\mathbf{x} = (0, 0, z)$  due to a point source at the origin in a homogeneous medium. According to the Gaussian beam summation method,<sup>7,8</sup> one first obtains the field at  $\mathbf{x}$  due to a Gaussian beam propagating in the direction  $\mathbf{e}_r$ . This is accomplished using the paraxial approximation about the central ray of the beam (see Fig. 1). The individual beam contribution is

$$u_\theta(\mathbf{x}) = \left( \frac{ik\epsilon}{8\pi^2} \right) \frac{1}{(\epsilon + s)} \exp \left[ ik \left( s + \frac{1}{2} \frac{n^2}{\epsilon + s} \right) \right], \quad (9)$$

where  $\epsilon$  is the arbitrary beam parameter,<sup>7,8</sup> subject only to the constraint that  $\text{Im } \epsilon < 0$ . An exhaustive discussion of the dependence of the beam method on  $\epsilon$  is given in Ref. 9. Also from Fig. 1,  $s = z \cos \theta$  and  $n = z \sin \theta$ . The pre-exponential amplitude in Eq. (9) is deduced from the requirement that the beam sum satisfy

$$\int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta u_\theta(\mathbf{x}) \sim \frac{e^{ikz}}{4\pi z} \quad (10)$$

asymptotically for  $k \gg 1$ . The upper limit to the  $\theta$  integral in Eq. (10) is unspecified, indicating that only a fan of Gaussian beams in the neighborhood of  $\theta = 0$  is required.

Now consider the three-dimensional (3D) point-source representation in Eq. (4) with the radius  $a$  complex. Referring to Fig. 1, the following paraxial approximation is valid for small angle  $\theta$ :

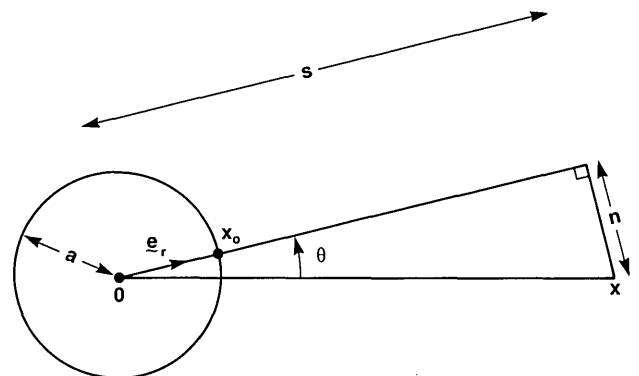


Fig. 1. A complex point source  $\mathbf{x}_0$  on the surface of the sphere of complex radius  $a$  and receiver at  $\mathbf{x}$ . The exact distance is  $R(\mathbf{x}, \mathbf{x}_0) \equiv [(s-a)^2 + n^2]^{1/2}$ . The paraxial approximation is in Eq. (11).

$$R(\mathbf{x}, a\mathbf{e}_r) = s - a + \frac{1}{2} \frac{n^2}{s - a} + O(\theta^4). \quad (11)$$

Let  $a = -\epsilon$  and substitute Eq. (11) into Eqs. (1) and (4) to obtain for the integrand in Eq. (4)

$$\frac{1}{4\pi j_0(ka)} u^G(\mathbf{x}, a\mathbf{e}_r) \sim \left[ \frac{e^{ik\epsilon}}{16\pi^2 j_0(k\epsilon)} \right] \frac{1}{(\epsilon + s)} \times \exp \left[ ik \left( s + \frac{1}{2} \frac{n^2}{\epsilon + s} \right) \right], \quad (12)$$

which is asymptotically valid for  $\theta \ll 1$ . Note the similarity between Eq. (9) and expression (12). The difference is in the amplitude factors. The exact amplitude of expression (12) reduces to the Gaussian beam amplitude asymptotically for

$$-k \operatorname{Im} \epsilon \gg 1, \quad (13)$$

in which case

$$\frac{e^{ik\epsilon}}{16\pi^2 j_0(k\epsilon)} \sim \frac{ik\epsilon}{8\pi^2}. \quad (14)$$

Thus, satisfaction of the inequality (13) is a necessary condition for the validity of the Gaussian beam summation method. The Gaussian beam method, therefore, has its origins in the exact complex point-source representation.

We also give the equations analogous to expressions (9)–(14) in two dimensions. Let  $\mathbf{x} = (x, 0)$ ,  $s = x \cos \phi$ , and  $n = x \sin \phi$ . Then the beam in direction  $\phi$  gives

$$u_\phi(\mathbf{x}) = \left( \frac{\sqrt{\epsilon}}{4\pi i} \right) \frac{1}{(\epsilon + s)^{1/2}} \exp \left[ ik \left( s + \frac{1}{2} \frac{n^2}{\epsilon + s} \right) \right], \quad (15)$$

where the amplitude follows from the asymptotic matching,

$$\int u_\phi(\mathbf{x}) d\phi \sim e^{-i3\pi/4} (8\pi kx)^{-1/2} e^{ikx}. \quad (16)$$

The right-hand side of expression (16) is the asymptotic approximation to Eq. (5). In comparison, the integrand of the exact representation Eq. (6) becomes, with  $a = -\epsilon$ , and, using a paraxial approximation similar to Eq. (11),

$$\frac{1}{2\pi J_0(ka)} u^G(\mathbf{x}, a\mathbf{e}_r) \sim \left[ \frac{\exp[i(k\epsilon - 3\pi/4)]}{2\sqrt{k}(2\pi)^{3/2} J_0(k\epsilon)} \right] \frac{1}{(\epsilon + s)^{1/2}} \times \exp \left[ ik \left( s + \frac{1/2 n^2}{\epsilon + s} \right) \right]. \quad (17)$$

The right-hand side of expression (17) reduces to  $u_\phi(\mathbf{x})$  when the asymptotic condition (13) is satisfied.

## A COMPARISON OF THE TWO METHODS BY EXAMPLE

The complex source and Gaussian beam methods are now compared by considering the problem of refraction through a hemispherical interface. The geometry of the configuration is illustrated in Fig. 2, where the point source is at S and the receiver R is on the axis through SO. The position of R is specified by the distance  $x$  from O. A typical ray diagram for the refracted real rays is shown in Fig. 3.

In order to use the complex point-source representation for the real point source at S, we must consider the interme-

diate problem of determining the field at R due to a typical complex point source,  $S_\theta$  in Fig. 2, for arbitrary complex  $\epsilon$ . One method is to use the exact solution for a point source outside a sphere that can be expressed as a sum of spherical harmonics and analytically extend it to account for  $S_\theta$  in complex space. Alternatively, and this is the method that we adopt here, at high frequencies  $\omega a/c_0 \gg 1$ , one can approximate the field of a complex point source using complex ray theory.<sup>2,4</sup> This is the analytical extension of Hamilton–Jacobi ray theory into complex space. For the present application, it can be thought of as an extension of Fermat’s principle of least travel time, only now the travel time from  $S_\theta$  to R is complex. The minimization procedure is described in Appendix A, where it is shown that the complex ray from  $S_\theta$  to R is refracted at P on the interface, where P can be parameterized by a complex value of the angle  $\psi$  (see Fig. 2). Details of the complex ray-tracing procedure and the summation are given in Appendix A.

The complex ray-tracing problem simplifies in the neighborhood of the real rays defined by the axes of the complex

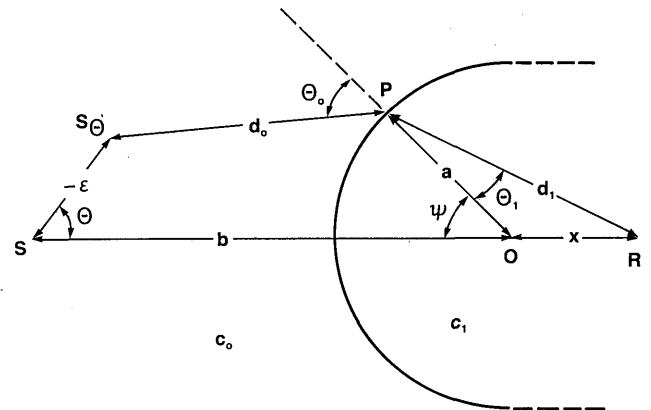


Fig. 2. Geometry of the point source at S, receiver at R, and spherical interface.  $S_\theta$  represents a typical complex point source on the complex sphere about S.

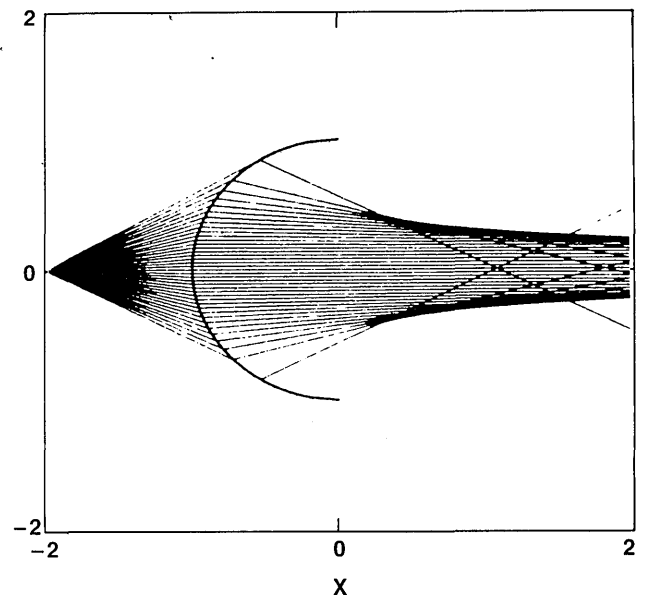


Fig. 3. The ray diagram for the spherical interface with  $a = 1$ ,  $b = 2$ ,  $c_0 = 2c_1$ .

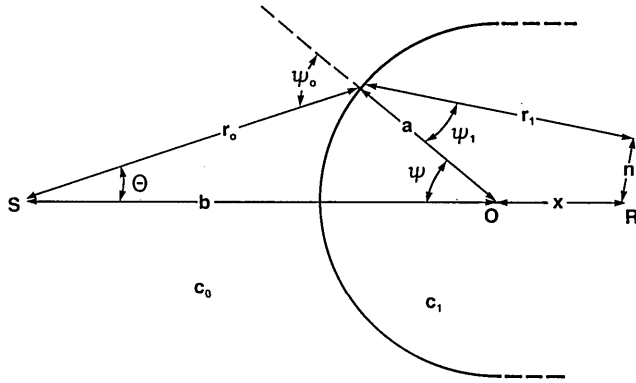


Fig. 4. The Gaussian beam geometrical parameter for the spherical interface.

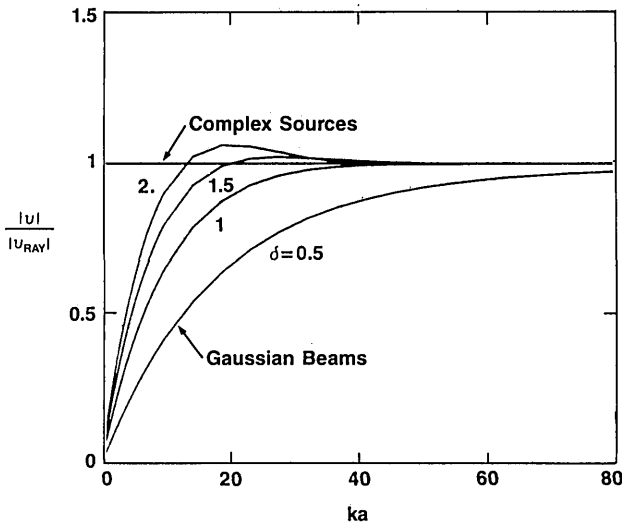


Fig. 5. Comparison of the Gaussian beam and complex point source methods for the spherical interface with  $c_0 = c_1$ ,  $b = 2a$ , and receiver at  $x = 0$ . Four values of  $\epsilon = -i\delta$  are considered.

point sources. These are the Gaussian beam rays associated with the complex sources. They remain in real space, and the parameters of the nearby rays in complex space can be obtained by paraxial approximation. This is essentially what the Gaussian beam summation method does. It replaces the difficult task of complex ray tracing by the much simpler one of real ray tracing. However, the price paid for the simplification is the paraxial approximation. The real ray paths for the Gaussian beam method applied to the interface problem are shown in Fig. 4. The details of the calculation are given in Appendix B.

The simplest ray theory approximation to the field at R considers only the direct ray from S to R through O. This yields

$$u_{\text{RAY}} = \frac{\exp\left\{ik\left[b - a + \frac{c_0}{c_1}(a + x)\right]\right\}}{2\pi r(1 + c_0/c_1)}, \quad (18)$$

where

$$r(x) = \frac{c_1}{c_0} \left\{ b - \frac{x}{a} \left[ \frac{c_0}{c_1}(b - a) - b \right] \right\}. \quad (19)$$

The solution in Eq. (18) has a focus at  $x = x_F$ , such that  $r(x_F)$

$= 0$ . The focus is at infinity for the parameters of Fig. 3. Also, there is a caustic along the central axis from  $x = x_C$  to  $x = x_F$ , where  $x_C < x_F$  is

$$x_C = b / \left[ \left( \frac{b^2}{a^2} - 1 \right)^{1/2} \left( \frac{c_0^2}{c_1^2} - 1 \right)^{1/2} - 1 \right]. \quad (20)$$

The value of  $x_C$  for the parameters of Fig. 3 is  $x_C = 1$ .

We first consider the degenerate case of  $c_1 = c_0$ . Then the problem reduces to one of a point source in an infinite homogeneous medium. The solution  $u_{\text{RAY}}$  of Eq. (18) is exact. Figure 5 shows the comparison of the complex source method and the Gaussian beam method for different values of the arbitrary parameter  $\epsilon$  as a function of frequency  $ka = \omega a/c_0$ . As expected, the complex source method is exact, independent of  $\epsilon$ . However, the Gaussian beam results are quite dependent on this "arbitrary" parameter. A thorough study of the dependence of the method on  $\epsilon$  is contained in Ref. 9, in which several canonical problems are considered. The configuration considered here is different from those in Ref. 9, but our findings are in qualitative agreement.

Next, let  $c_0 = 2c_1$  with the other parameters unchanged. Then the ray diagram is as shown in Fig. 3, with the focus at infinity. In this case the amplitude of  $u_{\text{RAY}}$  is independent of  $x$  and is equal to  $1/6\pi$ . Again, as Fig. 6 shows, the agreement between the complex source solution and  $u_{\text{RAY}}$  is excellent, although not exact. However, the maximum relative error was found to be less than 1%. The Gaussian beam method fares worse than for the homogeneous medium (Fig. 5) for a given value of the parameter  $\epsilon$ . One might expect that the beam solution would be better than in the homogeneous medium, because of the focusing of the rays, as shown in Fig. 3. The focusing should reduce the error incurred in using the paraxial approximation. However, as Ref. 9 illustrates by numerous examples, the dependence of the Gaussian beam results on  $\epsilon$  is often counterintuitive, especially when curved interfaces are involved.

The beam and complex source methods are compared in Fig. 7 for different receiver positions at a fixed frequency. The only significant error incurred by the complex source method was at  $x \sim 0.5$ , for  $\epsilon = -2i$ , where the complex root to the ray equation (A5) was found to be unstable. At larger values of  $x$  than shown here, the effects of the caustic  $x \geq 1$

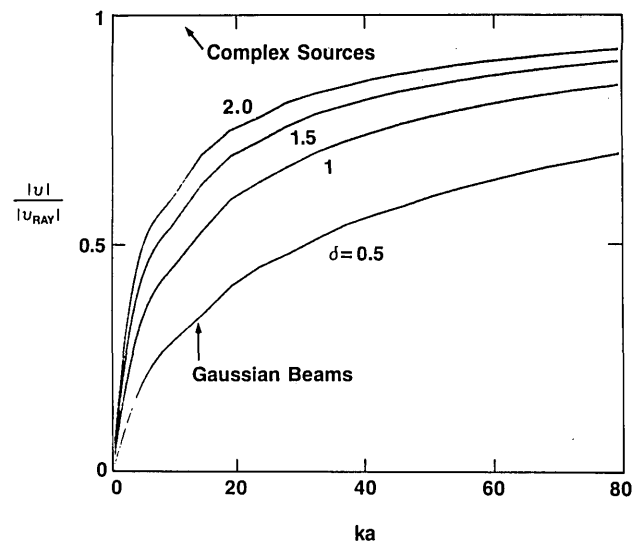


Fig. 6. The same as Fig. 5, except for  $c_0 = 2c_1$ .

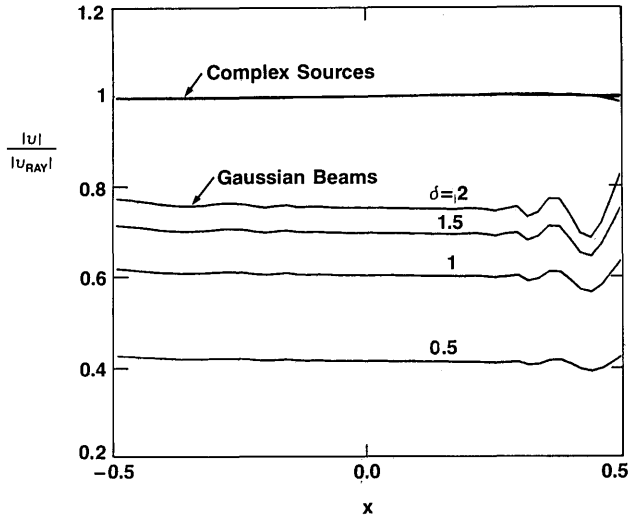


Fig. 7. Comparison of the Gaussian beam and complex point source methods for the spherical interface with  $c_0 = 2c_1$ ,  $b = 2$ ,  $a = 1$ ,  $ka = 20$ , and variable receiver positions on the axis. The absolute magnitude of  $u_{\text{RAY}}$  is  $1/6\pi$ , independent of  $x$ .

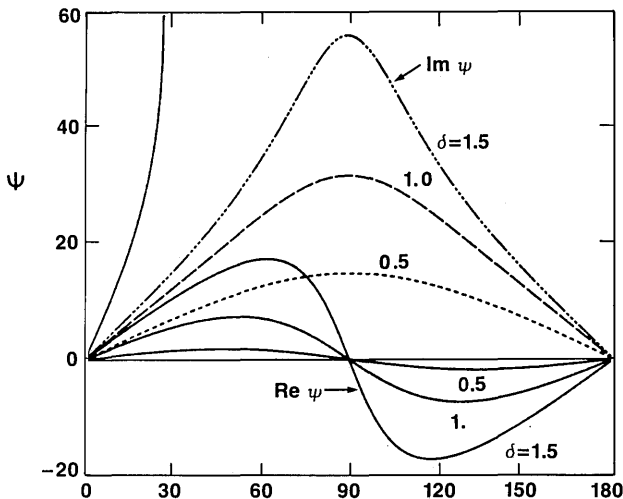


Fig. 8. The angle  $\psi$  of the point of refraction on the spherical interface (see Figs. 2 and 4). The upper-left-hand curve is for the Gaussian beams and is independent of  $\epsilon$ . The other curves show the real and imaginary parts of  $\psi$  for the complex point sources with  $\delta = i\epsilon$ . The parameters are  $a = 1$ ,  $b = 2$ ,  $c_0 = 2c_1$ .

can become significant. In this region, the single ray solution, Eq. (18), is invalid. Also, the complex ray solution in this region has more than one ray contribution from each source. In fact, there is always a multiplicity of complex rays for each source at a given value of  $x$ , but it is only for  $x \geq 1$  that those on the branches associated with the caustic become important. We defer until later a fuller discussion of the complex ray picture, since it is not germane to the present topic. However, we note that the results of Figs. 6 and 7 were also checked by using a Kirchhoff integral on the interface. This yielded the same solution as  $u_{\text{RAY}}$ , modulo spurious end point contributions that are difficult to remove. In general, the relative error of the Kirchhoff approximation was far greater than that of the complex source method.

Finally, we have plotted in Fig. 8 the real and imaginary parts of the complex angle  $\psi$  that defines the point P at

which the complex ray from  $S_\theta$  to R intersects the interface. Note that the imaginary part increases as the imaginary part of  $\epsilon$  increases in magnitude. The value of  $\psi$  follows from Eq. (A4). This may be solved for small  $\theta$  as

$$\psi = \frac{-\epsilon\theta}{b + \epsilon - \frac{c_0}{c_1}x \frac{(b + \epsilon - a)}{(a + x)}} + O(\theta^2). \quad (21)$$

Thus,  $\psi$  is generally complex for  $\theta \neq 0$ . The real angle  $\psi$  associated with the Gaussian beam rays from Eq. (B4) is also plotted for comparison. This has a cutoff at  $\theta = \sin^{-1}(a/b)$ , which is  $30^\circ$  in Fig. 8. Thus we see that the central rays of the beams do not intersect the complex interface at the correct points. However, the approximation (21) suggests a procedure whereby the implicit complex ray tracing of the Gaussian beams may be improved. In this method we analytically continue the interface into complex space using a quadratic or paraxial approximation to the interface about the point of intersection of the real ray SR. Thus the Gaussian beam rays can be shifted slightly into complex space, it is hoped in the right direction. We have tested this procedure on a similar problem in two dimensions and found dramatic improvements in the Gaussian beam results. However, it is not clear how to generalize this procedure to arbitrary configurations, so we will not pursue it further.

## CONCLUSIONS

A new representation has been given for a point source in terms of complex point sources. The representation can also be viewed as an exact basis for the Gaussian beam representation of a point source. From the latter viewpoint, it offers a similar but more rigorously justified approach to problems in inhomogeneous media. The alternative approach requires performing complex ray tracing rather than just ray tracing in real space. The comparison of the complex ray-tracing solution with that of the standard Gaussian beam method for the refraction problem considered shows that the former is highly accurate and independent of the arbitrary parameter  $\epsilon$ . However, it is questionable in general whether the accuracy and stability gained by using the complex ray-tracing approach would justify the additional complexity of the solution method.

## APPENDIX A: COMPLEX RAY TRACING THROUGH A SPHERICAL INTERFACE

Referring to Fig. 2, consider the point  $S_\theta$  on the complex sphere with center at S. The complex distance  $SS_\theta$  is  $-\epsilon$ . There is a point source at  $S_\theta$  and a receiver at R, which lies in real space. The wave speed is discontinuous across the surface of the sphere of radius  $a$ , centered at O. The refractive index of the medium at R is  $c_0/c_1$ . We restrict our attention to the field at R due to singly refracted rays; i.e., we neglect multiple reflections from the far side of the sphere. In this sense, the interface acts as a hemispherical lens.

One way to find the complex ray path from  $S_\theta$  to R is by finding the path in complex space that makes the travel time stationary—Fermat's principle. We complexify the interface by defining it as the complex surface

$$x^2 + y^2 + z^2 = a^2. \quad (A1)$$

Since the media on either side of the interface are homogeneous, the ray path is composed of two line segments  $S_\theta P$  and  $PR$  that intersect the interface at  $P$  (see Fig. 2). The point  $P$  may be parameterized by the complex angle  $\psi$ . The travel time from  $S_\theta$  to  $R$  is  $d_0/c_0 + d_1/c_1$ , where, referring to Fig. 2,

$$d_0(\psi) = [(b + \epsilon \cos \theta - a \cos \psi)^2 + (\epsilon \sin \theta + a \sin \psi)^2]^{1/2}, \quad (A2)$$

$$d_1(\psi) = (a^2 + x^2 + 2ax \cos \psi)^{1/2}. \quad (A3)$$

The stationary condition is  $d(d_0/c_0 + d_1/c_1)/d\psi = 0$ , which yields the following equation for  $\psi$ :

$$\frac{1}{d_0} [b \sin \psi + \epsilon \sin(\theta + \psi)] = \frac{c_0}{c_1} \frac{x}{d_1} \sin \psi. \quad (A4)$$

Once  $\psi$  is found as a function of  $\theta$ , the field at  $R$  due to  $S_\theta$  is

$$U(\theta) = \frac{T(\theta_0) \exp[i\omega(d_0/c_0 + d_1/c_1)]}{4\pi d_0(1 + d_1/D_1)^{1/2}(1 + d_1/D_2)^{1/2}}, \quad (A5)$$

where

$$\sin \theta_1 = \frac{x}{d_1} \sin \psi, \quad (A6)$$

$$\sin \theta_0 = \frac{c_0}{c_1} \sin \theta. \quad (A7)$$

The transmission coefficient is

$$T(\theta) = 2 \cos \theta / [\cos \theta + (c_0^2/c_1^2 - \sin^2 \theta)^{1/2}], \quad (A8)$$

and the focal distances  $D_1$  and  $D_2$  follow from standard geometrical optics (e.g., Ref. 10):

$$D_1 = \cos^2 \theta_1 / \left[ \frac{c_1}{c_0} \frac{\cos^2 \theta_0}{d_0} + \frac{1}{a} \left( \frac{c_1}{c_0} \cos \theta_0 - \cos \theta_1 \right) \right], \quad (A9)$$

$$D_2 = \left[ \frac{c_1}{c_0 d_0} + \frac{1}{a} \left( \frac{c_1}{c_0} \cos \theta_0 - \cos \theta_1 \right) \right]^{-1}. \quad (A10)$$

The integral of all the contributions from the points  $S_\theta$  on the complex sphere becomes, from Eq. (4),

$$u = \frac{1}{2j_0(k\epsilon)} \int_0^\pi \sin \theta U(\theta) d\theta. \quad (A11)$$

## APPENDIX B: THE GAUSSIAN BEAM SUMMATION METHOD

Referring to Fig. 4, the refracted beams are defined only for  $0 \leq \theta < \theta_m$ ,

$$\theta_m = \sin^{-1}(a/b). \quad (B1)$$

Within this range of angles, the various parameters in Fig. 4 are

$$\sin \psi_0 = \frac{b}{a} \sin \theta, \quad (B2)$$

$$\sin \psi_1 = \frac{c_1}{c_0} \sin \psi_0, \quad (B3)$$

$$\psi = \psi_0 - \theta, \quad (B4)$$

$$r_0 = b \cos \theta - a \cos \psi_0, \quad (B5)$$

$$r_1 = a \cos \psi_1 + x \cos(\psi_1 - \psi), \quad (B6)$$

$$n = a \sin \psi_1 + x \sin(\psi_1 - \psi). \quad (B7)$$

The radii of curvature of the refracted beam follow from standard wave-front curvature equations (e.g., Ref. 10):

$$z_1 = \cos^2 \psi_1 / \left[ \frac{c_1}{c_0} \frac{\cos^2 \psi_0}{(r_0 + \epsilon)} + \frac{1}{a} \left( \frac{c_1}{c_0} \cos \psi_0 - \cos \psi_1 \right) \right], \quad (B8)$$

$$z_2 = \left[ \frac{c_1}{c_0} \frac{1}{(r_0 + \epsilon)} + \frac{1}{a} \left( \frac{c_1}{c_0} \cos \psi_0 - \cos \psi_1 \right) \right]^{-1}. \quad (B9)$$

The individual Gaussian beam contribution is

$$V(\theta) = \frac{ik}{8\pi^2} \frac{T(\psi_0) \exp \left\{ ik \left[ r_0 + \frac{c_0}{c_1} r_1 + \frac{1}{2} \frac{c_0}{c_1} n^2 / (r_1 + z_1) \right] \right\}}{(1 + r_0/\epsilon)(1 + r_1/z_1)^{1/2}(1 + r_1/z_2)^{1/2}}, \quad (B10)$$

where  $k = \omega/c_0$  and  $T$  is defined in Eq. (A8). The Gaussian beam summation integral is then

$$u = 2\pi \int_0^{\theta_m} V(\theta) \sin \theta d\theta. \quad (B11)$$

## ACKNOWLEDGMENTS

This research was supported, in part, by Exxon Research and Engineering Company. The support of the Rutgers University Research Council is appreciated. Particular thanks to B. White and R. Burridge.

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