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The influence of beam type on the back reflection of ultrasonic beams from a liquid–solid interface

Andrew N. Norris
Exxon Research and Engineering Company, Annandale, New Jersey 08801

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We consider the phenomenon of back reflection of a beam of ultrasound from a liquid–solid interface. We show that the magnitude of the back reflection depends critically upon the profile of the incident beam. The beam profile is first related to the excitation field at the plane of the transducer. Different beams are considered by changing the field at the transducer. We show that a purely Gaussian profile gives no back reflection. Other beam profiles with less severe decay away from the beam center can produce an effect of the same order as observed experimentally. We present an example of such a beam.

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INTRODUCTION

It has been observed in experiments[1] that a bounded beam of ultrasound when incident upon a liquid–solid interface produces a cone of diffracted ultrasound, in addition to the normal specular reflection. The generator of the cone is normal to the interface and its semi-angle is the Rayleigh critical angle. The phenomenon is generally referred to as back reflection because pulse-echo measurements of a beam incident at the critical angle show that energy appears to be reflected from the interface back towards the transducer.[2] Recent experimental observations of de Billy *et al.*[3,4] have provided quantitative proof of the existence of this phenomenon.

In a previous paper,[5] which we shall refer to as I, we attempted to explain the phenomenon for the special case of a Gaussian beam. The Gaussian profile was chosen because it was thought to model accurately the experimental beams[6,7] and also because it had proved to be convenient in understanding the Schoch displacement effect.[8] However, subsequent comparison[9] showed the Gaussian beam theory of I to produce a miniscule effect, far too small in magnitude to be of physical significance. The same result has been shown by Stamnes.[10]

In this letter we show that the magnitude of the back reflection depends critically upon the beam profile. The Gaussian beam is, in a sense, the "worst case" beam type. Other beam profiles can possibly produce an effect which is quantitatively in agreement with experiment. We consider one such beam profile in detail.

I. THE BACK REFLECTED BEAM

Consider an arbitrary two-dimensional beam incident upon the interface. The scattered field at an observation point (x,z) in the liquid $z<0$ is

$$u(x,z)=\frac{1}{2\pi}\int_{-\infty}^{\infty} R(k_x)U(k_x)\exp\left[i(k_x x-k_z z)\right]dk_x, \tag{1}$$

where $U(k_x)$ is the transform of the incident field at $z=0$

$$U(k_z)=\int_{-\infty}^{\infty} u^i(x,0)\exp(-ik_x x)dx. \tag{2}$$

In I we took u^i to be Gaussian, but we now leave it unspecified. We assume that the incident beam direction has a positive x component. The pole of the reflection coefficient $R(k_x)$ at $k_x=k_p\equiv k\sin\theta_p$ is responsible for the production of leaky waves propagating in the positive x direction. The complex angle θ_p has a real part equal to the Rayleigh critical angle and a small imaginary part. The pole at $k_x=-k_p$ produces leaky waves in the negative x direction. Let

$x = -r\sin\theta$ and $z = -r\cos\theta$. If $kr \gg 1$ and $U(k_x)$ is slowly varying near $k_x = -k_p$, the contribution to u from the pole at $k_x = -k_p$ is approximately u^B, where

$$u^B(r,\theta) = U(-k_p)\mathrm{Res}[R(-k_p)] \times \frac{1}{2\pi}\int_C \exp[ikr\cos(\nu-\theta)]\frac{\cos\nu\, d\nu}{\sin\theta_p - \sin\nu}. \quad (3)$$

Here Res is the residue and the contour C goes from $\nu = -\pi/2 + i\infty$ to $-\pi/2$, then to $\pi/2$, and finally to $\pi/2 + i\infty$. From Eq. (60) of I we have

$$u^B(r,\theta) \sim (-i/2)\cos\theta_p U(-k_p) \times \mathrm{Res}[R(-k_p)]\exp[ikr\cos(\theta-\theta_p)] \times \mathrm{erfc}[e^{-i\pi/4}(2kr)^{1/2}\sin[(\theta_p-\theta)/2]]. \quad (4)$$

The discussion in I, particularly the results of Fig. 5, show that u^B behaves like a back reflected leaky wave beam. Its magnitude is of the order of $|U(-k_p)|$. This is the crucial quantity which we must estimate.

II. BEAM MODELS

Consider a semi-infinite liquid half-space $-\infty < X < \infty$, $Z < 0$, with acoustic field $v(X,Z)$. We assume that the half-width w_0 on $Z = 0$ is much larger than the wavelength, or $kw_0 \gg 1$. Then, if $v(X,0)$ is specified as $v(X,0) = f(X)$, this high-frequency assumption implies that $\partial v(X,0)/\partial Z \simeq -ikv(X,0) = -ikf(X)$. The high-frequency assumption is justified *a priori* from Sec. IV. $f(X)$ is symmetric about $X = 0$ and tends to zero faster than $X^{-1/2}$ as $X \to \infty$. With $X = -R\sin\Theta$, $Z = -R\cos\Theta$, the farfield, $kR \gg 1$, is given by

$$v(R,\Theta) = [e^{ikR}/(kR)^{1/2}]E(\Theta), \quad (5)$$

where the directivity function E is

$$E(\Theta) = [e^{-i\pi/4}/(8\pi)^{1/2}]k(1+\cos\Theta)F(k\sin\Theta) \quad (6)$$

and

$$F(k\sin\Theta) = \int_{-\infty}^{\infty} f(X)\exp(ikX\sin\Theta)\, dX. \quad (7)$$

For example, if

$$f(X) = A\int_{-a}^{a}\exp[-(X-Y)^2/w_0^2]\, dY = \frac{\pi^{1/2}}{2}w_0 A\left[\mathrm{erfc}\left(\frac{X-a}{w_0}\right) - \mathrm{erfc}\left(\frac{X+a}{w_0}\right)\right], \quad (8)$$

then

$$E(\Theta) = \frac{e^{-i\pi/4}}{\sqrt{2}}w_0 A(1+\cos\Theta)\frac{\sin(ka\sin\Theta)}{\sin\Theta} \times \exp[-(\tfrac{1}{2}kw_0\sin\Theta)^2]. \quad (9)$$

This particular choice of $f(X)$ includes two well-known beam patterns as limits. First let $w_0 \to 0$, keeping $Aw_0\sqrt{\pi} = 1$. Then $f(x) = 1$ for $|X| < a$, 0 for $|X| < a$. The radiation pattern is

$$E(\Theta) = \frac{e^{-i\pi/4}}{(2\pi)^{1/2}}(1+\cos\Theta)\frac{\sin(ka\sin\Theta)}{\sin\Theta} \quad (10)$$

which is $1+\cos\Theta$ times the sinc pattern. Second, let $a \to 0$, keeping $2Aa = 1$. Then $f(X) = \exp(-X^2/w_0^2)$ and

$$E(\Theta) = (e^{-i\pi/4}/\sqrt{8})\, kw_0(1+\cos\theta)\exp[-(\tfrac{1}{2}kw_0\sin\Theta)^2] \quad (11)$$

which represents a Gaussian profile. In general, Eqs. (8) and (9) represent a hybrid sinc–Gaussian beam. The Gaussian profile of Eq. (10) is special in that it has no sidelobes, which are characteristic of the sinc pattern. The general profile of Eq. (9) has sidelobes, but they disappear as a vanishes. Also, as w_0 increases the beam half-width decreases, so that any sidelobes will have exponentially small magnitude.

As a second example, consider the beam defined by the function

$$f(x) = 1/[1+(X/w_0)^2]. \quad (12)$$

The beam pattern follows from Eqs. (6), (7), and (12) as

$$E(\Theta) = (\pi/8)^{1/2}e^{-i\pi/4}kw_0(1+\cos\Theta)\exp(-kw_0|\sin\Theta|). \quad (13)$$

We will return to this beam profile later.

III. ESTIMATE OF THE BACK REFLECTION STRENGTH

Consider the configuration in Fig. 1. The transducer is in the plane $Z = 0$. We assume that the beam is in the farfield of the transducer (the Fraunhofer region) as it intersects the interface at $z = 0$. The angle of incidence θ_i is arbitrary, though we will specialize later to the case of back reflection at the Rayleigh critical angle. On the interface near the beam center, we have

$$R \sim d + x\sin\theta_i, \quad (14)$$

$$\Theta \sim (x/d)\cos\theta_i. \quad (15)$$

We assume that Θ is very small over the beamwidth on the interface, so that $\sin\Theta \sim \Theta$ there. The incident field on the interface then follows from Eqs. (5), (14), and (15) as

$$u^i(x,0) \sim \frac{\exp[ik(d + x\sin\theta_i)]}{(kd)^{1/2}}E\left(\frac{x}{d}\cos\theta_i\right). \quad (16)$$

Then by Eqs. (2), (6), (7), and (16),

$$|U(-k_p)| = \left(\frac{\pi d}{2k}\right)^{1/2}\left|f\left[d\left(\frac{\sin\theta_i + \sin\theta_p}{\cos\theta_i}\right)\right]\right| \quad (17)$$

$$\times[1+\sec\theta_i].$$

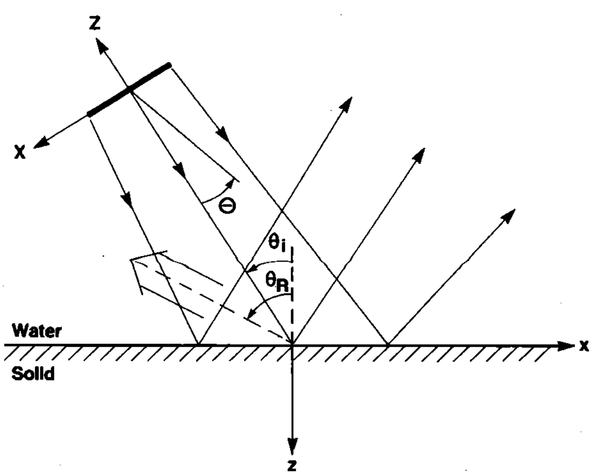

FIG. 1. Schematic of the transducer–interface configuration. The distance d is the distance from the center of the transducer ($X = Z = 0$) to the point at which the beam center hits the interface ($x = z = 0$). In practice, d is usually much greater than the beamwidth on the interface.

Equation (17) relates the back reflection strength to the transducer function f.

IV. RESULTS AND DISCUSSION

We now consider a typical experimental configuration. We use the data of de Billy *et al.*[3] From Ref. 3 we have $k = 19\ \mathrm{mm}^{-1}$ and $\theta_i = 31°$, which is the Rayleigh critical angle for water/stainless steel. The half-beamwidth on the interface is 12 mm. We also need to know the distance d from the transducer. This parameter is not specified in Ref. 3, but we take $d = 385$ mm to agree with the presumed value in Ref. 10. This is in fact a typical acoustic distance for this type of experiment; Breazale *et al.*[7] report a value for d of 400 mm. Now, our analysis has assumed the interface to be in the farfield of the transmitter. This is true if $ka^2/d < 1$, where a is the radius of the transmitter. In Refs. 3 and 7 transducers with values of $a = 3.5$ mm were used, for which $ka^2/d \sim 0.6$.

If we assume the incident beam to be Gaussian, then by Eqs. (11) and (16), for the half-beamwidth to be 12 mm on the interface the half-width on the transducer must be $w_0 \sim 4$ mm. We note that this value is hardly consistent with the requirement that the transducer radius be at least as great as w_0, raising doubts as to whether the beam is really Gaussian. Now, if we compute $|U(-k_p)|$ of Eq. (17) with $f(X) = \exp(-X^2/w_0^2)$, we get an incredibly small number. This cannot possibly agree with the experimental observations that the back reflection strength is of the order of 40 dB below the specular reflection strength. We have previously presented this conclusion regarding the unfeasibility of the Gaussian beam.[9] The same conclusion has also been reached by Stamnes.[10]

Next we consider the general profile defined by Eq. (9), which includes the Gaussian profile as a special case. No matter what value of a we take, the requirement that the profile have half-width of 12 mm on the interface again implies that $w_0 \sim 4$ mm. Then computing $|U(-k_p)|$ of Eq. (17) with $f(X)$ of Eq. (8) we again get a negligible result.

If instead, we assume the transducer profile is as in Eq. (12), then Eqs. (13) and (16) imply $w_0 \sim 2$ mm. Referring to Eqs. (12) and (17), we now have

$$f\{d[(\sin\theta_i + \sin\theta_p)/\cos\theta_i]\} \sim 0.2 \times 10^{-4} \tag{18}$$

which gives a value of $|U(-k_p)|$, and hence of the back reflection strength, that is more in line with experimental observations.

In practice, the beam profiles encountered in experiments are not purely Gaussian. The Gaussian profile is popular because it is a good approximation to most beam types near the beam center, and it does not possess sidelobes. However, the back reflection effect depends upon the beam spectral component at $k_x = -k_p$, which has little to do with the profile near the beam center. Therefore, we must know the beam pattern far from its center in order to calculate the back reflection. The Gaussian profile has a very rapid decay away from its center but the profile of Eq. (12) has a much less severe decay. This difference accounts for the markedly different results obtained for the two profiles.

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